3

Notions of Copulas

In this chapter, we introduce the notion of copulas, which describes the dependence between several random variables. These variables can be the returns of different assets or the value of a given asset at different times, and more generally, any set of economic variables. We present some examples of classical families of copulas and provide several illustrations of the usefulness of copulas for actuarial,[1] economic, and financial applications.

Until relatively recently, the correlation coefficient was the measure of reference used to quantify the amplitude of dependence between two assets. From the old hypothesis or belief that the marginal distribution of returns is Gaussian, it was natural to extend this assumption of normality to the multivariate domain. Recall that only under the assumption of multivariate normality[2] is the correlation coefficient necessary and sufficient to capture the full dependence structure between asset returns. The growing attacks of the past three decades and the now overwhelming evidence against the Gaussian hypothesis also cast doubts on the relevance of the correlation coefficient as an adequate measure of dependence. See for instance [404] for a specific test of multivariate normality of asset returns. Actually, it is now clear that the correlation coefficient is grossly insufficient to provide an accurate description of the dependence between two assets [64, 148, 149] and that it is necessary to characterize the full joint multivariate distribution of asset returns. This is all the more important for rare large events whose deviations from normality are the most extreme both in amplitude and dependence.

Consider for simplicity the problem of characterizing the bivariate distribution of the returns of only two assets. It is essential to realize that the bivari-

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[1] Actuarial science is a sister discipline of statistics. Actuaries play an important role in many of the financial plans that involve people, *e.g.*, life insurance, pension plans, retirement benefits, car insurance, unemployment insurance, and so on.

[2] To some extent, the correlation coefficient also adequately quantifies the dependence between elliptically distributed random variables, even if it may yield spurious conclusions – especially in the far tails – as we shall see in the next chapters.

ate distribution embodies two qualitatively different pieces of information on the two assets. On the one hand, it contains the two marginal distributions; on the other hand, it contains information on the dependence between the two assets irrespective of their individual (marginal) distributions. Only the introduction of the copula allows one to operate a clean dissection between these two pieces of information. The role of the copula of two random variables is precisely to offer a complete and unique description of the dependence structure existing between them, excluding all (parasiting) information on the marginal distribution of the random variables.

Such an approach in terms of copulas has witnessed a recent burst of interest and of activity spurred by its practical and theoretical implications. From an applied view point, determining the dependence between assets is at the core of risk management: the dependence governs (i) the optimization of diversification of risks by aggregation in portfolios, (ii) the hedging strategies based on derivatives, and (iii) the securitization[3] of different risky instruments to sell them to third parties. Specifically, the advantage of the copula formulation is to provide a better understanding and quantification of the interactions between assets by determining the diverse dependence structures between the various sources of risk. Applications to finance include the calculation of VaR (Value-at-Risk) and portfolio optimization [145], the calculation of option prices [99, 112], and credit risk [184, 186]. For various applications to insurance, see [110, 183, 478].

From a fundamental viewpoint, it is reasonable to think that the structure of dependence between assets reflects the underlying mechanisms at work in financial markets. In particular, the dependence between assets is in part the result of the interactions between the agents investing in the stock market[4]. Not only are investors responsible for the individual variations and fluctuations of assets but, by their asset allocation choices (buying or selling such or such security rather than another), they also create dependence between assets. It can thus be hoped that the study of the dependence between assets may complement the understanding of the important mechanisms at work in stock markets and therefore of the interactions between agents. It should also help in narrowing down the relevant macroscopic parameters influencing investors in their asset allocation.

Before presenting copulas and their fundamental properties, we should stress that this body of results applies also when the structure of dependence is time-varying. This remark is important since there is *a priori* no principle or reason for the dependence to be constant [380, 409, 412]. One should thus

[3] The process of aggregating similar instruments, such as loans or mortgages, into a negotiable security.

[4] Of course, the observed dependence between assets has also other inputs than just the action of economic agents on financial markets. The macroeconomic variables also play an important role, especially for assets belonging to the same economic sector, which are collectively sensitive to the same variations of the macroeconomic landscape. This is the stance taken by factor models described in Chap. 1.

study its dynamics in addition. However, such a study of the time-dynamics of the multivariate dependence structure between assets is extremely delicate both from an empirical and theoretical point of view. In addition, as we show in Chap. 6, some apparent time-varying dependence may appear as a spurious consequence of conditioning the measures of dependence on market phases with large volatility, for instance. This mechanism appears to explain a large part of the empirical observations on time-varying dependence, suggesting that it would be sufficient to model the time-dependent properties of volatility alone. We thus make the simplifying assumption that any possible time-dependence of the statistical properties of assets is entirely embedded in the evolution of the marginal distributions of their returns, while the dependence structure between assets remains invariant.

3.1 What is *Dependence*?

The notion of independence of random variables is very easy to define. From elementary probability theory, two random variables X and Y are *independent* if and only if, for any x and y in the supports of the distributions,

$$\Pr[X \leq x; Y \leq y] = \Pr[X \leq x] \cdot \Pr[Y \leq y] \ , \tag{3.1}$$

or equivalently

$$\Pr[X \leq x \mid Y] = \Pr[X \leq x] \ . \tag{3.2}$$

In other words, two random variables are independent if the knowledge of a piece of information about one of the random variables does not bring any new insight on the other one.

The notion of *dependence* is much more subtle to define, or at least to quantify. Let us start with the concept of *mutual complete dependence* [290]. It seems natural that two real random variables X and X' are mutually completely dependent if the knowledge of X implies the knowledge of X', and reciprocally. This statement simply means that there exists a one-to-one mapping f such that:

$$X' = f(X), \quad \text{almost everywhere} \ , \tag{3.3}$$

which, as stressed in [270], implies the perfect predictability of one of the random variables from the other one. The mapping f is either strictly increasing or strictly decreasing. In the first case, the random variables are said to be *comonotonic*.

In a second stage of our investigation of the concept of dependence, let us ask what could be the meaning of the following statement:

> The random variables X and Y exhibit the same dependence as the random variables X' and Y'.

A possible interpretation, explored in this chapter, is that the random variables X and X', on the one hand, and Y and Y', on the other hand, are comonotonic. In this case, all variables or functions describing the dependence between two (and more generally several) random variables should enjoy the property of invariance under an arbitrary increasing mapping. Let us assume that there exists a function C describing the dependence of the random variables X and Y and a function C' describing the dependence of the random variables X' and Y'. Writing that X and X' (respectively Y and Y') are comonotonic,

$$X' = h_1(X) , \tag{3.4}$$
$$Y' = h_2(Y) , \tag{3.5}$$

where h_1 and h_2 are increasing functions on $\mathbb{R}$ (if we consider real-valued random variables), the property of invariance under strictly increasing mapping reads $C = C'$.

Let us now show how to build C. Does the usual correlation coefficient qualify? While the correlation coefficient measures some kind of dependence, it is only able to account for a linear dependence.[5] Therefore, it does not fulfill the requirement for a general concept of dependence which should involve any nonlinear monotonic structure. Thus, we must look for something else.

Let us consider the two random variables X and Y and their joint distribution function denoted by H:

$$H(x,y) = \Pr[X \le x; Y \le y] . \tag{3.6}$$

The marginal distributions of X and Y are respectively:

$$F(x) = \Pr[X \le x] = \lim_{t\to\infty} H(x,t) , \tag{3.7}$$
$$G(y) = \Pr[Y \le y] = \lim_{t\to\infty} H(t,y) . \tag{3.8}$$

For simplicity, let us assume that F and G are continuous and increasing, so that the usual inverses F^{-1} and G^{-1} exist. Then, let us define

$$C(u,v) = H\left(F^{-1}(u), G^{-1}(v)\right) , \quad \forall u,v \in [0,1] . \tag{3.9}$$

Let us now focus on the random variables X' and Y' given by (3.4–3.5) above. It is clear that their joint distribution function is

$$\begin{aligned} H'(x,y) &= \Pr[X' \le x; Y' \le y] = \Pr\left[X \le h_1^{-1}(x); Y \le h_2^{-1}(y)\right] \\ &= H\left(h_1^{-1}(x), h_2^{-1}(y)\right) , \end{aligned} \tag{3.10}$$

[5] Indeed, consider the linear regression $Y = \beta X + \epsilon$ where β is a constant and X and ϵ are two independently distributed centered random variables with variances $\mathrm{Var}(X)$ and $\mathrm{Var}(\epsilon)$ respectively. Then, the knowledge of the covariance $\mathrm{Cov}(X,Y)$ and of the variance $\mathrm{Var}(X)$ of X is equivalent to the knowledge of the linear dependence between X and Y: $\mathrm{Cov}(X,Y) = \beta\, \mathrm{Var}(X)$. The correlation coefficient, $\mathrm{Corr}(X,Y) \equiv \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} = \left[1 + \mathrm{Var}(\epsilon)/(\beta^2 \mathrm{Var}(X))\right]^{-1/2}$, involves in addition an information on $\mathrm{Var}(\epsilon)$.

while their marginal distributions are:

$$F'(x) = \Pr[X' \le x] = F\left(h_1^{-1}(x)\right) , \tag{3.11}$$
$$G'(y) = \Pr[Y' \le y] = G\left(h_2^{-1}(y)\right) . \tag{3.12}$$

Now, considering

$$C'(u,v) = H'\left(F'^{-1}(u), G'^{-1}(v)\right) , \quad \forall u,v \in [0,1] , \tag{3.13}$$

elementary algebraic manipulations show that

$$C(u,v) = C'(u,v), \quad \forall u,v \in [0,1] . \tag{3.14}$$

It turns out that the function C defined by (3.9) is the only object obeying the property of invariance under strictly increasing mapping and which entirely captures the full dependence between X and Y.

The following properties follow from simple calculations:

- $C(u,1) = u$ and $C(1,v) = v$, $\forall u,v \in [0,1]$,
- $C(u,0) = C(0,v) = 0$, $\forall u,v \in [0,1]$,
- C is 2-increasing, namely, for all $u_1 \le u_2$ and $v_1 \le v_2$:

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0 . \tag{3.15}$$

This last property is a simple translation of the nonnegativity of probabilities, specifically of the following expression:

$$\Pr\left[F^{-1}(u_1) \le X \le F^{-1}(u_2); G^{-1}(v_1) \le Y \le G^{-1}(v_2)\right] \ge 0 . \tag{3.16}$$

As we shall see in the sequel, these three properties define the mathematical object called *copula*, which has been introduced by A. Sklar in the late 1950s [443] in order to describe the general dependence properties of random variables.

3.2 Definition and Main Properties of Copulas

This section provides a brief survey of the main properties of copulas, emphasizing the most important definitions and theorems useful in the following. For exhaustive and general presentations, we refer to [248, 370] and to [74, 183] for introductions oriented to financial and actuarial applications.

The definition of a copula of n random variables generalizes the intuitive definition (3.9) presented above for the bivariate copula.

Definition 3.2.1 (Copula). *A function $C : [0,1]^n \longrightarrow [0,1]$ is a n-copula if it enjoys the following properties :*

- $\forall u \in [0,1]$, $C(1,\dots,1,u,1\dots,1) = u$,
- $\forall u_i \in [0,1]$, $C(u_1,\dots,u_n) = 0$ *if at least one of the u_i's equals zero ,*

- *C is grounded and n-increasing, i.e., the C-volume of every box whose vertices lie in $[0,1]^n$ is positive.*

It is clear from this definition that a copula is nothing but a multivariate distribution with support in $[0,1]^n$ and with uniform marginals. It immediately follows that a convex sum of copulas remains a copula. The fact that such mathematical objects can be very useful for representing multivariate distributions with arbitrary marginals has been suggested in the previous introductory section and is stated more formally in the following result [443].

Theorem 3.2.1 (Sklar's Theorem). *Given a n-dimensional distribution function F with* continuous[6] *(cumulative) marginal distributions $F_1, \ldots, F_n$, there exists a unique n-copula $C : [0,1]^n \longrightarrow [0,1]$ such that:*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \ . \tag{3.17}$$

Thus, the copula combines the marginals to form the multivariate distribution. This theorem provides both a parameterization of multivariate distributions and a construction scheme for copulas. Indeed, given a multivariate distribution F with marginals $F_1, \ldots, F_n$, the function

$$C(u_1, \ldots, u_n) = F\left(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\right) \tag{3.18}$$

is automatically an n-copula.[7] This copula is the copula of the multivariate distribution F. We will use this method in the sequel to derive the expressions of standard copulas such as the Gaussian copula or the Student's copula.

In addition to the copula itself, it is often very useful to consider the two following quantities:

Definition 3.2.2. *Given n random variables $X_1, \ldots, X_n$ with marginal* survival *distributions $\bar{F}_1, \ldots, \bar{F}_n$ and joint* survival *distribution $\bar{F}$, the* survival copula *$\bar{C}$ is such that:*

$$\bar{C}\left(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)\right) = \bar{F}(x_1, \ldots, x_n) \ . \tag{3.19}$$

The dual copula C^* of the copula C of $X_1, \ldots, X_n$ is defined by:

$$C^*(u_1, \ldots, u_n) = 1 - \bar{C}(1-u_1, \ldots, 1-u_n) \ , \quad \forall u_1, \ldots, u_n \in [0,1] \ . \tag{3.20}$$

[6] When this assumption fails, Sklar's theorem still holds, but in a weaker sense: a representation like (3.17) still exists but is not unique anymore.

[7] The quantile function, or generalized inverse, F_i^{-1} of the distribution F_i can be defined by:

$$F_i^{-1}(u) = \inf\{x \mid F_i(x) \geq u\}, \quad \forall u \in (0,1).$$

When the distribution function F_i is strictly increasing, F_i^{-1} denotes the usual inverse of F_i. In fact, any quantile function can be chosen. But, for noncontinuous margins, the copula (3.18) depends upon the precise quantile function which is selected.

While the survival copula is indeed a true copula, the dual copula is not. However, it can be simply related to the probability that (at least) one of the X_i's is less than or equal to x_i. Indeed, one can easily check that:

$$\Pr\left[\bigcup_{i=1}^{n}\{X_i \leq x_i\}\right] = C^*\left(F_1(x_1), \ldots, F_n(x_n)\right) \ . \tag{3.21}$$

A very powerful property shared by all copulas is their invariance under arbitrary increasing mapping of the random variables (this has been shown for the case of the bivariate copulas in the derivation ending with (3.14)):

Theorem 3.2.2 (Invariance Theorem). *Consider n continuous random variables $X_1, \ldots, X_n$ with copula C. Then, if $h_1(X_1), \ldots, h_n(X_n)$ are increasing on the ranges of $X_1, \ldots, X_n$, the random variables $Y_1 = h_1(X_1), \ldots, Y_n = h_n(X_n)$ have exactly the same copula C.*

Let us stress again that this result demonstrates that the full dependence between the n random variables is completely captured by the copula, independent of the shape of the marginal distributions. In other words, the Invariance Theorem shows that the copula is an intrinsic measure of dependence between random variables. Under a monotonic change of variable from an old variable to a new variable, these two variables are comonotonic by definition. Intuitively, as explained in the previous section, it is natural that a measure of dependence between two random variables should be insensitive to the substitution of one of the variables by a comonotonic variable: if X and X' are two comonotonic variables, one expects the same dependence structure for the pair (X,Y) and for the pair (X',Y). This is precisely the content of the Invariance Theorem on copulas. In contrast, a measure of dependence such as the correlation coefficient which is function of both the copula and the marginal distribution is not invariant under a monotonic change of variable. It does not constitute an intrinsic measure of dependence (we will come back in detail on this point in Chap. 4). The benefit of using copulas is the decoupling between the marginal distribution and the dependence structure, which justifies the separate study of marginal distributions on the one hand and of the dependence on the other hand.

Let us now state several useful properties enjoyed by copulas. First, any copula is uniformly continuous:

Proposition 3.2.1. *Given an n-copula C, for all $u_1, \ldots, u_n \in [0,1]$ and all $v_1, \ldots, v_n \in [0,1]$:*

$$|C(v_1, \ldots, v_n) - C(u_1, \ldots, u_n)| \leq |v_1 - u_1| + \cdots + |v_n - u_n| \ . \tag{3.22}$$

This result is a direct consequence of the property that copulas are n-increasing. Indeed, restricting ourselves to the bivariate case for the simplicity of the exposition, the triangle inequality implies

$$|C(v_1, v_2) - C(u_1, u_2)| = |C(v_1, v_2) - C(u_1, v_2) + C(u_1, v_2) - C(u_1, u_2)| \\ \leq |C(v_1, v_2) - C(u_1, v_2)| + |C(u_1, v_2) - C(u_1, u_2)|,$$

and by (3.15), with some of the arguments put equal to 0 or 1, we have

$$|C(v_1, v_2) - C(u_1, v_2)| \leq |v_1 - u_1| , \tag{3.23}$$

and

$$|C(u_1, v_2) - C(u_1, u_2)| \leq |v_2 - u_2| , \tag{3.24}$$

which leads to the expected result.

Besides, it follows that a copula is differentiable almost everywhere:

Proposition 3.2.2. *Let C be an n-copula. For almost all $(u_1, \ldots, u_n) \in [0,1]^n$, the partial derivative of C with respect to u_i exists and:*

$$0 \leq \frac{\partial C}{\partial u_i}(u_1, \ldots, u_n) \leq 1 . \tag{3.25}$$

These two properties show that copulas enjoy nice regularity (or smoothness) conditions. In fact, the later one will turn out to be very useful for numerical simulations, as we shall see in Sect. 3.5.

Due to the property that copulas are n-increasing, we can find an upper and a lower bound for any copula. Choosing $u_2 = v_2 = 1$ in (3.15), we obtain that any bivariate copula satisfies

$$C(u, v) \geq u + v - 1 . \tag{3.26}$$

Since, in addition, a copula is non-negative, we obtain a lower bound for any bivariate copula:

$$C(u, v) \geq \max(u + v - 1, 0) . \tag{3.27}$$

Similarly, choosing alternatively $(u_1 = 0, v_2 = 1)$ and $(u_2 = 1, v_1 = 0)$, we get an upper bound for any bivariate copula

$$C(u, v) \leq \min(u, v) . \tag{3.28}$$

It is clear that these two bounds fulfill all the requirements of copulas, qualifying the functions $\max(u + v - 1, 0)$ and $\min(u, v)$ as genuine bivariate copulas. These two bounds are thus the tightest possible bounds. Generalization to higher dimension is straightforward, so that we can state

Proposition 3.2.3 (Fréchet-Hoeffding Upper and Lower Bounds). *Given an n-copula C, for all $u_1, \ldots, u_n \in [0,1]$:*

$$\max(u_1 + \ldots + u_n - n + 1, 0) \leq C(u_1, \ldots, u_n) \leq \min(u_1, \ldots, u_n) . \tag{3.29}$$

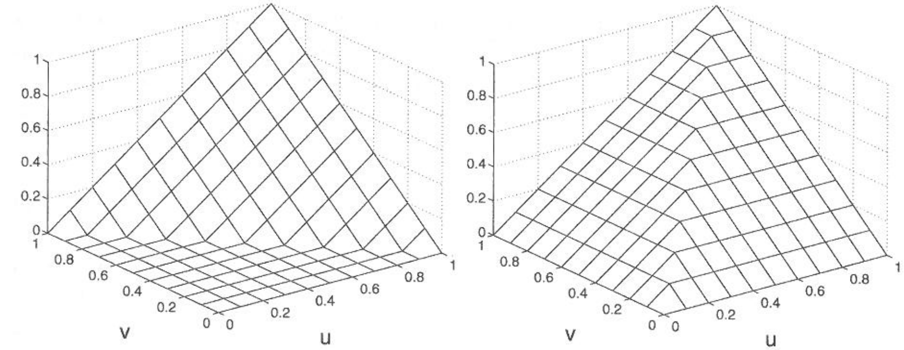

Fig. 3.1. The Fréchet-Hoeffding lower (*left panel*) and upper (*right panel*) bounds for bivariate copulas

These lower and upper bounds, which constitute the so-called *Fréchet-Hoeffding bounds*, are represented in Fig. 3.1 for the bivariate case. The upper bound is itself an n-copula, while the lower one is a copula only for $n = 2$. However, this lower bound remains the best possible insofar as, for any fixed point $(u_1, \ldots, u_n) \in [0,1]^n$, there exists a copula $\tilde{C}$ such that, at this particular point:

$$\tilde{C}(u_1, \ldots, u_n) = \max(u_1 + \cdots + u_n - n + 1, 0) . \tag{3.30}$$

The Fréchet-Hoeffding upper bound represents the strongest form of dependence that several random variables can exhibit. In fact, it is nothing but the copula associated with comonotonicity. Similarly, when $n = 2$, the Fréchet-Hoeffding lower bound is nothing but the copula of countermonotonicity.

3.3 A Few Copula Families

As shown from Sklar's theorem 3.2.1, for each multivariate distribution, one can easily derive a copula. Notwithstanding their formidable number, a few copula families play a more important role.

3.3.1 Elliptical Copulas

Elliptical copulas derive from multivariate elliptical distributions [252]. Here, we give the two most important examples, the Gaussian and Student's copulas. By construction, these two copulas are close to each other in their central part, and become closer and closer in their tail only when the number of degrees of freedom of the Student's copula increases. As a consequence, it is sometimes difficult to distinguish between them, even with sensitive tests. However, as we shall see in Chap. 4, these two copulas may have drastically different behaviors with respect to the dependence between extremes.

Multiplicative factor models, which account for most of the stylized facts observed on financial time series, generate distributions with elliptical copulas. Multiplicative factor models contain in particular multivariate stochastic volatility models with a common stochastic volatility factor. They can be formulated as

$$\mathbf{X} = \sigma \cdot \mathbf{Y} , \tag{3.31}$$

where σ is a positive random variable modeling the volatility, $\mathbf{Y}$ is a Gaussian random vector, independent of σ and $\mathbf{X}$ is the vector of assets returns. In this framework, the multivariate distribution of asset returns $\mathbf{X}$ is an elliptical multivariate distribution. For instance, if the inverse $1/\sigma^2$ of the square of the volatility σ is a constant times a χ^2-distributed random variable with ν degrees of freedom, the distribution of asset returns will be the Student distribution with ν degrees of freedom. When the volatility follows ARCH or GARCH processes, then the asset returns are also elliptically distributed with fat-tailed marginal distributions. Such elliptical multivariate distribution ensures that each asset X_i is asymptotically distributed according to a regularly varying distribution:[8] $\Pr\{|X_i| > x\} \sim L(x) \cdot x^{-\nu}$ – where $L(\cdot)$ denotes a slowly varying function – with the same exponent ν for all assets.

Elliptical copulas have the advantage of being easily synthesized numerically, which makes their use convenient for numerical simulations and for the study of scenarios. This results from the fact that it is easy to generate Gaussian or Student's distributed random variables which, upon appropriate monotonic changes of variables, give the correct marginal distributions while conserving the copula unchanged.

The Gaussian Copula

The Gaussian copula is the copula derived from the multivariate Gaussian distribution. The Gaussian copula provides a natural setting for generalizing Gaussian multivariate distributions into so-called meta-Gaussian distributions. Meta-Gaussian distributions have been introduced in [283] (see [163] for a generalization to meta-elliptical distributions) and have been applied in many areas, from the analysis of experiments in high-energy particle physics [265] to finance [453]. These meta-Gaussian distributions have exactly the same dependence structure as the Gaussian distributions while differing in their marginal distributions which can be arbitrary.

Let Φ denote the standard Normal (cumulative) distribution and $\Phi_{\rho,n}$ the n-dimensional standard Gaussian distribution with correlation matrix $\boldsymbol{\rho}$. Then, the Gaussian n-copula with correlation matrix $\boldsymbol{\rho}$ is

$$C_{\rho,n}(u_1, \ldots, u_n) = \Phi_{\rho,n}\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)\right) , \tag{3.32}$$

[8] See footnote 3 page 39.

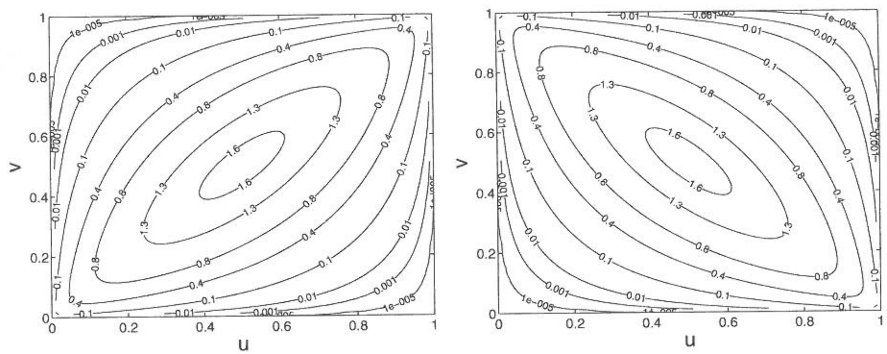

Fig. 3.2. Contour plot of the density (3.34) of the bivariate Gaussian copula with a correlation coefficient $\rho = 0.8$ (*left panel*) and $\rho = -0.8$ (*right panel*)

whose density (see Fig. 3.2)

$$c_{\rho,n}(u_1, \ldots, u_n) = \frac{\partial C_{\rho,n}(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n} \tag{3.33}$$

reads

$$c_{\rho,n}(u_1, \ldots, u_n) = \frac{1}{\sqrt{\det \boldsymbol{\rho}}} \exp\left(-\frac{1}{2} \boldsymbol{y}^t(\boldsymbol{u})(\boldsymbol{\rho}^{-1} - \mathrm{Id})\boldsymbol{y}(\boldsymbol{u})\right) \tag{3.34}$$

with $\boldsymbol{y}^t(\boldsymbol{u}) = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n))$. Note that Theorem 3.2.1 and equation (3.18) ensure that $C_{\rho,n}(u_1, \ldots, u_n)$ in (3.32) is a copula.

The Gaussian copula is completely determined by the knowledge of the correlation matrix $\boldsymbol{\rho}$. The parameters involved in the description of the Gaussian copula are simple to estimate, as we shall see in Chap. 5.

Student's Copula

Student's copula is derived from Student's multivariate distribution. It provides a natural generalization of Student's multivariate distributions, in the form of meta-elliptical distributions [163]. These meta-elliptical distributions have exactly the same dependence structure as the Student's distributions while differing in their marginal distributions which can be arbitrary.

Given an n-dimensional Student distribution $T_{n,\rho,\nu}$ with ν degrees of freedom and a shape matrix $\boldsymbol{\rho}$[9]

$$T_{n,\rho,\nu}(\mathbf{x}) = \frac{1}{\sqrt{\det \boldsymbol{\rho}}} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{n/2}} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{d\mathbf{x}}{\left(1 + \frac{\mathbf{x}^t \rho^{-1} \mathbf{x}}{\nu}\right)^{\frac{\nu+n}{2}}} , \tag{3.35}$$

[9] Note that the shape matrix $\boldsymbol{\rho}$ is nothing but the correlation matrix when the number of degrees of freedom ν is larger than 2, namely when the second moments of the variables X_i's exist.

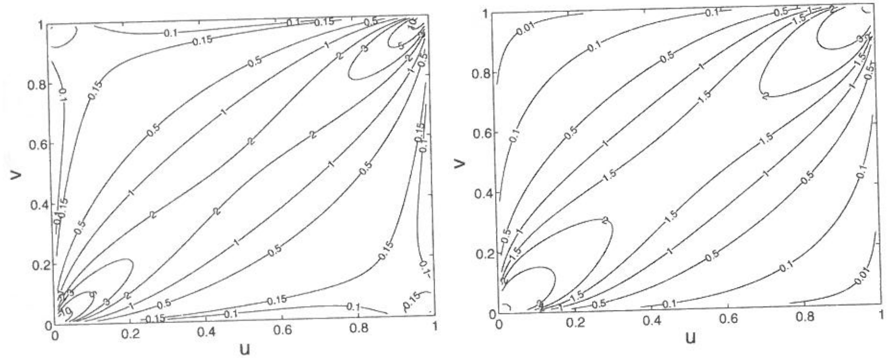

Fig. 3.3. Contour plot of the density (3.37) of a bivariate Student t copula with a shape parameter $\rho = 0.8$ and $\nu = 2$ degrees of freedom (*left panel*) or $\nu = 10$ degrees of freedom (*right panel*). For small ν's, the difference between the Student copula and the Gaussian copula is striking on both diagonals. As ν increases, this difference decreases on the second diagonal but remains large (for $\nu = 10$) on the main diagonal, as can be observed by comparing the above right with the left panel of Fig. 3.2

the corresponding Student's copula reads:

$$C_{n,\rho,\nu}(u_1, \dots, u_n) = T_{n,\rho,\nu}\left(T_\nu^{-1}(u_1), \dots, T_\nu^{-1}(u_n)\right) , \tag{3.36}$$

where T_ν is the univariate Student's distribution with ν degrees of freedom. The density of the Student's copula is thus

$$c_{n,\rho,\nu}(u_1, \dots, u_n) = \frac{1}{\sqrt{\det \rho}} \frac{\Gamma\left(\frac{\nu+n}{2}\right)\left[\Gamma\left(\frac{\nu}{2}\right)\right]^{n-1}}{\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^n} \frac{\prod_{k=1}^n \left(1 + \frac{y_k^2}{\nu}\right)^{\frac{\nu+1}{2}}}{\left(1 + \frac{y^t \rho^{-1} y}{\nu}\right)^{\frac{\nu+n}{2}}} , \tag{3.37}$$

where $y^t = (T_\nu^{-1}(u_1), \dots, T_\nu^{-1}(u_n))$. See also Fig. 3.3.

Since Student's distribution tends to the normal distribution when ν goes to infinity, Student's copula tends to the Gaussian copula as $\nu \to +\infty$ [350]:

$$\sup_{u \in [0,1]^n} |C_{n,\rho,\nu}(u) - C_{\rho,n}(u)| \longrightarrow 0, \quad \text{as } \nu \to +\infty . \tag{3.38}$$

The description of a Student copula relies on two parameters: the shape matrix ρ, as in the Gaussian case, and in addition the number of degrees of freedom ν. An accurate estimation of the parameter ν is rather difficult and this can have an important impact on the estimated value of the shape matrix.[10] As a consequence, the Student's copula may be more difficult to calibrate and to use than the Gaussian copula.

3.3.2 Archimedean Copulas

The importance of this class of copulas lies in that it encompasses a very large number of copulas while enjoying a certain number of interesting properties. In addition, as pointed out by Frees and Valdez [183], a large number of models developed to account for the dependence between various sources of risks in the theory of insurance lead to Archimedean copulas. The factor models constitute, however, a notable exception. While linear factor models play a fundamental role in the phenomenological description of interactions between financial assets, Archimedean copulas are not adequate to describe their corresponding dependence structure. In the same vein, the Gaussian and Student's copulas, as well as any elliptical copula, are not Archimedean.

An Archimedean copula is defined as follows:

Definition 3.3.1 (Archimedean Copula). *Let φ be a continuous strictly decreasing, convex, function from $[0,1]$ onto $[0,\infty]$ and such that $\varphi(1) = 0$. Let $\varphi^{[-1]}$ be the pseudo-inverse of φ :*

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & \text{if } 0 \le t \le \varphi(0) , \\ 0, & \text{if } t \ge \varphi(0) , \end{cases} \tag{3.39}$$

then the function

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) \tag{3.40}$$

is an Archimedean copula with generator φ.

The generalization to an n-copula seems straightforward:

$$C_n(u_1, \dots, u_n) = \varphi^{[-1]}(\varphi(u_1) + \dots + \varphi(u_n)) . \tag{3.41}$$

However, this formulation holds— *i.e.*, C_n is actually an n-Archimedean copula – if and only if $\varphi^{[-1]}$ is n-monotonic:

$$(-1)^k \frac{d^k \varphi^{[-1]}(t)}{dt^k} \ge 0, \quad \forall k = 0, 1, \dots, n . \tag{3.42}$$

When this later relation holds for all $n \in \mathbb{N}$, $\varphi^{[-1]}$ is said *completely monotonic*. In such a case, the bivariate Archimedean copula can be generalized to any dimension.

[10] Lindskog *et al.* [307] have recently introduced a robust estimation technique for the calibration of the shape matrix of any elliptical copula, which is described in Chap. 5.

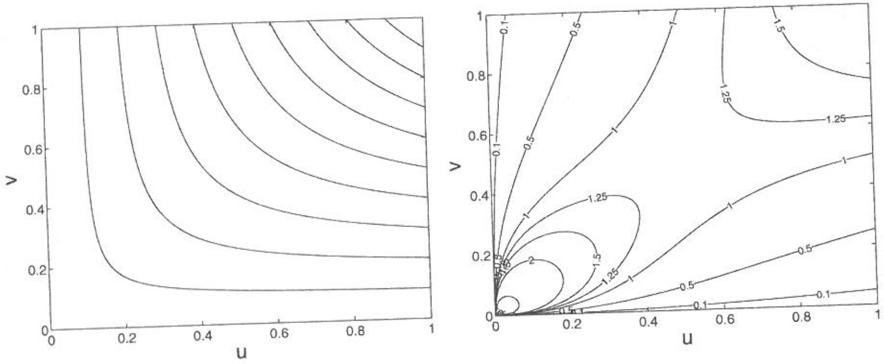

Fig. 3.4. Contour plot of Clayton's copula (*left panel*) and contour plot of its density (*right panel*) for parameter value $\theta = 1$

The complexity of the dependence structure between n variables usually described by a function of n variables is reduced and embedded, for Archimedean copulas, into the function of a single variable, the generator φ. This transforms a multidimensional formulation into a much simpler one-dimensional one.

Among the large number of copulas in the Archimedean family, the following copulas can be mentioned:

- Clayton's copula, which plays the role of a limit copula (see (3.61)):

$$C_\theta^{Cl}(u,v) = \max\left(\left[u^{-\theta} + v^{-\theta} - 1\right]^{-1/\theta}, 0\right) , \quad \theta \in [-1, \infty) \tag{3.43}$$

  with generator $\varphi(t) = \frac{1}{\theta}\left(t^{-\theta} - 1\right)$,
- Gumbel's copula, which plays a special role in the description of dependence using extreme value theory (see next Sect. 3.3.3):

$$C_\theta^{G}(u,v) = \exp\left(-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right) , \quad \theta \in [1, \infty) \tag{3.44}$$

  with generator $\varphi(t) = (-\ln t)^\theta$,
- Frank's copula:

$$C_\theta^{F}(u,v) = -\frac{1}{\theta} \ln\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right), \quad \theta \in \mathbb{R} \tag{3.45}$$

  with generator $\varphi(t) = -\ln \frac{e^{-\theta t} - 1}{e^{-\theta} - 1}$.

Note that the bivariate Fréchet-Hoeffding lower bound is an Archimedean copula, while the upper bound copula is not. For an overview of the members of the Archimedean family, we refer to Table 4.1 in [370].

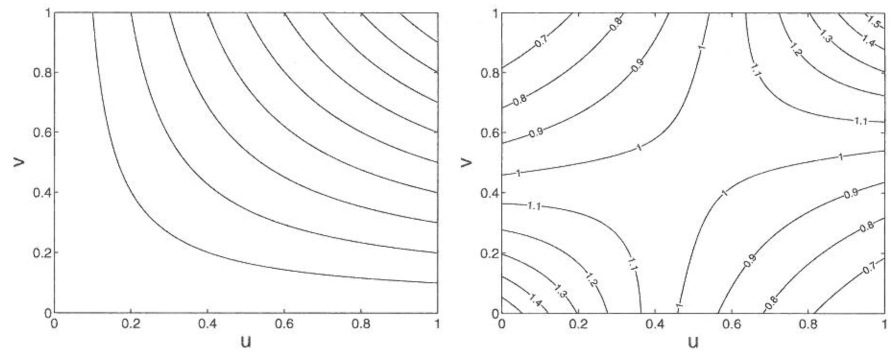

Fig. 3.5. Contour plot of Frank's copula (*left panel*) and contour plot of its density (*right panel*) for parameter value $\theta = 1$

A general procedure for constructing generators of the Archimedean copula has been proposed by Marshal and Olkin [348]. They have proved that, given a distribution function F defined on $\mathbb{R}^+$ such that $F(0) = 0$, the inverse $\varphi(t) = \phi^{-1}(t)$ of the Laplace transform of F

$$\phi(t) = \int_0^\infty e^{-t \cdot x} \, dF(x) \tag{3.46}$$

is the generator of an Archimedean copula.

This suggests that frailty models [236, 480] can provide a natural mechanism for generating random variables with Archimedean copulas. Such models are common in actuarial science, because they offer a simple way to study the joint mortality of a group of individuals sharing common risk factors (see [103, 182, 237] among many others). In finance, they can also model the joint distribution of defaults of different obligators subjected to the same set of economic factors.

In each case, one focuses on the continuous random variables T_i representing the survival time of the ith individual or company, *i.e.*, the time before death or default. Their individual survival distributions are defined by

$$S_i(t) = \Pr[T_i > t] , \tag{3.47}$$

with hazard rate:

$$h_i(t) = -\frac{d}{dt} \ln S_i(t) . \tag{3.48}$$

Conditional on a p-dimensional random vector Z representing the risk factors, one can use a proportional hazard model [111], with the i^{th} conditional hazard rate given by

$$h_i(t|Z) = e^{\beta \cdot Z} b_i(t) , \tag{3.49}$$

where the $b_i(t)$'s are the base-line hazard rates and $\boldsymbol{\beta}$ is the vector of regression parameters (the same for all individuals).

Defining the frailty variable $V = e^{\boldsymbol{\beta} \cdot Z}$ and integrating the conditional hazard rates, one obtains the expression of the conditional survival distributions:

$$S_i(t|V = v) = e^{-v \cdot f_i(t)}, \quad \text{where} \quad f_i(t) = \int_0^t b_i(s) ds \, . \tag{3.50}$$

Then, assuming that V has the distribution function F with Laplace transform ϕ (cf. (3.46)), the joint survival distribution of the T_i's is given by

$$\begin{aligned} \Pr[T_1 > t_1, \ldots, T_n > t_n] &= \mathrm{E}^V [S_1(t_1|V) \cdots S_n(t_n|V)], \\ &= \mathrm{E}^V \left[e^{-V \cdot (f_1(u_1) + \cdots + f_n(u_n))} \right], \\ &= \int_0^\infty e^{-v \cdot (f_1(u_1) + \cdots + f_n(u_n))} \, dF(v), \\ &= \varphi^{-1}(f_1(u_1) + \cdots + f_n(u_n)) \, . \end{aligned} \tag{3.51}$$

Since the unconditional marginal survival function of a given T_i reads

$$S_i(t_i) = E^V [S_i(t_i|V)] = \varphi^{-1}(f_i(u_i)) \, , \tag{3.52}$$

Sklar's theorem shows that the (survival) copula of all the T_i's is:

$$\bar{C}(u_1, \ldots, u_n) = \varphi^{-1}(\varphi(u_1) + \cdots + \varphi(u_n)) \, , \tag{3.53}$$

which is Archimedean, as expected.

As an example, let us consider Clayton's copula. Equation (3.43) shows that its generator is $\varphi(t) = t^{-\theta} - 1$, so that $\phi(t) = (1+t)^{-1/\theta}$, which is precisely the Laplace transform of a Gamma distribution $\Gamma(\theta^{-1}, 1)$ with parameter $1/\theta$, $\theta > 0$. As a consequence, considering a frailty variable V following a Gamma distribution with parameters $(1/\theta, 1)$, $\theta > 0$ and n conditionally independent random variables $U_i|V$, with conditional law:

$$\Pr[U_i \le u_i | V = v] = e^{-v \cdot (u_i^{-\theta} - 1)}, \quad u_i \in [0,1] \, , \tag{3.54}$$

one obtains n uniformly distributed random variable U_i, whose dependence structure is the Clayton copula with parameter θ.

Archimedean copulas enjoy the important property of *associativity*:

$$C_3(u, v, w) = C_2(u, C_2(v, w)) = C_2(C_2(u, v), w) \, , \tag{3.55}$$

where C_2 and C_3 respectively denote the bivariate and trivariate form of the copula under consideration. This property derives straightforwardly from (3.41). In other words, given three random variables U, V and W, the dependence between the first two random variables taken together and the third one alone is the same as the dependence between the first random variable taken

alone and the two last ones together. Therefore, if the dependence of the three random variables is described by an Archimedean copula, this implies a strong symmetry between the different variables in that they are exchangeable. As a consequence, when there is no reason to expect a breaking of symmetry between the random variables, an Archimedean copula may be a good choice to model their dependence. Such an assumption is often used in modeling large credit baskets. *A contrario*, when the random variables play very different roles, namely when they are not exchangeable, Archimedean copulas do not provide valid models of their dependence.

Another interesting property of Archimedean copulas is that their values $C(u,u)$ on the first bisectrix verify the following inequality:

$$C(u,u) < u, \quad \text{for all } u \in (0,1) \, . \tag{3.56}$$

Reciprocally, one can demonstrate [370, Theorem 4.1.6] that any copula possessing these two properties (associativity and $C(u,u) < u$) are Archimedean. This provides an intuitive understanding of the nature of Archimedean copulas. It also allows one to understand why the Fréchet–Hoeffding upper bound copula is not Archimedean. Indeed, although it enjoys the associativity property, the Fréchet-Hoeffding upper bound is such that $C(u,u) = u$ for all $u \in [0,1]$ (note that it is the only copula with this property).

Archimedean copulas obey an important limit theorem [260] of the type of the Gnedenko-Pikand-Balkema-de Haan (GPBH) theorem (see Chap. 2). Consider two random variables, X and Y, distributed uniformly on $[0,1]$, and whose dependence structure can be described by an Archimedean copula C. Then, the copula associated with the distribution of left-ordered quantiles tends, in most cases, to Clayton's copula (3.43) in the limit where the probability level of the quantiles goes to zero. To be more specific, let us denote by φ the generator of the copula C, *assumed differentiable*. Let us define the conditional distribution

$$F_u(x) = \Pr[X \le x | X \le u, \; Y \le u] = \frac{C(x \wedge u, u)}{C(u,u)}, \quad \forall x \in [0,1] \, , \tag{3.57}$$

where $x \wedge u$ means the minimum of x and u, and the conditional copula

$$\begin{aligned} C_u(x,y) &= \Pr[X \le F_u^{-1}(x), Y \le F_u^{-1}(y) | X \le u, \; Y \le u] \\ &= \frac{C\left(F_u^{-1}(x), F_u^{-1}(y)\right)}{C(u,u)}. \end{aligned} \tag{3.58}$$

One can first show that, provided that φ is a strick generator (that is, $\varphi(0)$ is infinite such that $\varphi^{[-1]} = \varphi^{-1}$), C_u is a strict Archimedean copula with generator:

$$\varphi_u(t) = \varphi\left(F_u^{-1}(t)\right) - \varphi(u), \tag{3.59}$$

$$= \varphi\left(t \cdot \varphi^{-1}(2\varphi(u))\right) - 2\varphi(u) \, , \tag{3.60}$$

from which, it follows that the limiting behavior of C_u, as u goes to zero, is:

$$\lim_{u \to 0} C_u(x,y) = C_\theta^{Cl}(x,y), \quad \forall (x,y) \in [0,1] \times [0,1] , \tag{3.61}$$

provided that φ is regularly varying[11] at zero, with index $\theta \in \mathbb{R}_+$. When $\theta = 0$, C_u tends to the independent copula while it tends to the Fréchet-Hoeffding upper bound copula when $\theta = \infty$.

Thus, Clayton's copula plays, in some sense, a role similar in n dimensions to the generalized Pareto distribution in one dimension:

$$G_\xi(x) = 1 - (1 + \xi \cdot x)^{-1/\xi} . \tag{3.62}$$

This result is of particular relevance in the study of multivariate statistics of extremes.

3.3.3 Extreme Value Copulas

Another family of copulas which is of common use is that of extreme value copulas. These copulas are derived from the dependence structure of multivariate generalized extreme value (GEV) distributions, which provide the limit distributions of the component-wise maxima of n-dimensional random vectors, after a suitable normalization.

Consider T iid n-dimensional random vectors $\boldsymbol{X_k} = (X_{k,1}, \ldots, X_{k,n})$, $k = 1, \ldots, T$ with distribution function F, and their component-wise maxima

$$M_{j,T} = \max_{1 \le k \le T} X_{k,j} . \tag{3.63}$$

For suitably chosen norming sequences $(a_{k,T}, b_{k,T})$, the limit distribution

$$\lim_{T \to \infty} \Pr \left(\frac{M_{1,T} - b_{1,T}}{a_{1,T}} \le z_1, \ldots, \frac{M_{n,T} - b_{n,T}}{a_{n,T}} \le z_n \right)$$

$$= \lim_{T \to \infty} F^T \left(a_{1,T} \cdot z_1 + b_{1,T}, \ldots, a_{n,T} \cdot z_n + b_{n,T} \right) , \tag{3.65}$$

if it exists, is given by

$$C \left(H_{\xi_1}(z_1), \ldots, H_{\xi_n}(z_n) \right) , \tag{3.66}$$

where H_ξ is a GEV distribution (see Chap. 2), and C is – by definition – an extreme value copula. Therefore, accounting for the general representation of multivariate extreme value (MEV) distributions (see [107]), we can state that:

[11] See footnote 3 page 39.

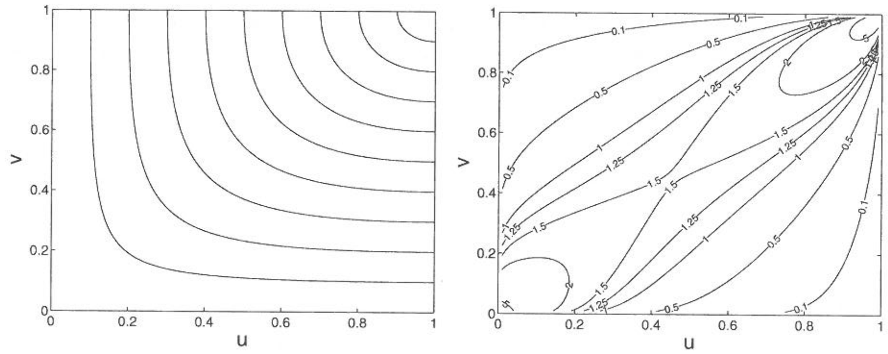

Fig. 3.6. Contour plot of Gumbel's copula (*left panel*) and of its density (*right panel*) for the parameter value $\theta = 2$

Definition 3.3.2 (Extreme Value Copula). *Any copula which admits the representation:*

$$C(u_1, \ldots, u_n) = \exp \left[-V \left(-\frac{1}{\ln u_1}, \ldots, -\frac{1}{\ln u_n} \right) \right] , \tag{3.67}$$

with

$$V(x_1, \ldots, x_n) = \int_{\Pi_n} \max_i \left(\frac{w_i}{x_i} \right) dH(w) , \tag{3.68}$$

where H is any positive finite measure such that $\int_{\Pi_n} w_i \, dH(w) = 1$ and Π_n is the (n-1)-dimensional unit simplex:

$$\Pi_n = \left\{ w \in \mathbb{R}_+^n : \sum_{i=1}^n w_i = 1 \right\} , \tag{3.69}$$

is an extreme value copula.

One immediately observes that V is a homogeneous function of degree -1. Thus, any extreme value copula satisfies [248]:

$$C \left(u_1{}^\alpha, \ldots, u_n{}^\alpha \right) = \left[C(u_1, \ldots, u_n) \right]^\alpha , \tag{3.70}$$

for all $u \in [0,1]^n$ and all $\alpha > 0$.

It is now easy to check that Gumbel's copula (3.44) belongs to the class of extreme value copula. It is depicted in Fig. 3.6: apart from a slight asymmetry with respect to the second bisectrix, it looks similar to a Student's copula.

The Fréchet-Hoeffding upper bound copula is also an extreme value copula since

$$\min \left(u_1{}^\alpha, \ldots, u_n{}^\alpha \right) = \min \left(u_1, \ldots, u_n \right)^\alpha . \tag{3.71}$$