Uncertainty and the Evaluation of Public Investment Decisions

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The implications of uncertainty for public investment decisions remain controversial. The essence of the controversy is as follows. It is widely accepted that individuals are not indifferent to uncertainty and will not, in general, value assets with uncertain returns at their expected values. Depending upon an individual's initial asset holdings and utility function, he will value an asset at more or less than its expected value. Therefore, in private capital markets, investors do not choose investments to maximize the present value of expected returns, but to maximize the present value of returns properly adjusted for risk. The issue is whether it is appropriate to discount public investments in the same way as private investments.

There are several positions on this issue. The first is that risk should be discounted in the same way for public investments as it is for private investments. It is argued that to treat risk differently in the public sector will result in overinvestment in this sector at the expense of private investments yielding higher returns. The leading proponent of this point of view is Jack Hirshleifer.[1] He argues that in perfect capital markets, investments are discounted with respect to both time and risk and that the discount rates obtaining in these markets should be used to evaluate public investment opportunities.

A second position is that the government can better cope with uncertainty than private investors and, therefore, government investments should not be evaluated by the same criterion used in private markets. More specifically, it is argued that the government should ignore uncertainty and behave as if indifferent to risk. The government should then evaluate investment opportunities according to their present value computed by discounting the expected value of net returns, using a rate of discount equal to the private rate appropriate for investments with certain returns. In support of this position it is argued that the government invests in a greater number of diverse projects and is able to pool risks to a much greater extent than private investors.[2] Another supporting line of argument is that many of the uncertainties which arise in private capital markets are related to what may be termed moral hazards. Individuals involved in a given transaction may hedge against the possibility of fraudulent behavior on the part of their associates. Many such risks are not present in the case of public investments and, therefore, it can be argued that it is not appropriate for the government to take these risks into account when choosing among public investments.

There is, in addition, a third position on the government's response to uncertainty. This position rejects the notion that indi-

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[1] J. Hirshleifer (1965, 1966) and Hirshleifer, J. C. De Haven, and J. W. Milliman (pp. 139–50).

[2] For this point of view, see P. A. Samuelson and W. Vickrey.

vidual preferences as revealed by market behavior are of normative significance for government investment decisions, and asserts that time and risk preferences relevant for government action should be established as a matter of national policy. In this case the correct rules for action would be those established by the appropriate authorities in accordance with their concept of national policy. The rate of discount and attitude toward risk would be specified by the appropriate authorities and the procedures for evaluation would incorporate these time and risk preferences. Two alternative lines of argument lead to this position. First, if one accepts the proposition that the state is more than a collection of individuals and has an existence and interests apart from those of its individual members, then it follows that government policy need not reflect individual preferences. A second position is that markets are so imperfect that the behavior observed in these markets yields no relevant information about the time and risk preferences of individuals. It follows that some policy as to time and risk preference must be established in accordance with other evidence of social objectives. One such procedure would be to set national objectives concerning the desired rate of growth and to infer from this the appropriate rate of discount.[3] If this rate were applied to the expected returns from all alternative investments, the government would in effect be behaving as if indifferent to risk.

The approach taken in this paper closely parallels the approach taken by Hirshleifer, although the results differ from his. By using the state-preference approach to market behavior under uncertainty, Hirshleifer demonstrates that investments will not, in general, be valued at the sum of the expected returns discounted at a rate appropriate for investments with certain returns.[4] He then demonstrates that using this discount rate for public investments may lead to non-optimal results, for two reasons. First, pooling itself may not be desirable.[5] If the government has the opportunity to undertake only investments which pay off in states where the payoff is highly valued, to combine such investments with ones that pay off in other states may reduce the value of the total investment package. Hirshleifer argues that where investments can be undertaken separately they should be evaluated separately, and that returns should be discounted at rates determined in the market. Second, even if pooling were possible and desirable, Hirshleifer argues correctly that the use of a rate of discount for the public sector which is lower than rates in the private sector can lead to the displacement of private investments by public investments yielding lower expected returns.[6]

For the case where government pooling is effective and desirable, he argues that rather than evaluate public investments differently from private ones, the government should subsidize the more productive private investments. From this it follows that to treat risk differently for public as opposed to private investments would only be justified if it were impossible to transfer the advantages of government pooling to private investors. Therefore, at most, the argument for treating public risks differently than private ones in evaluating investments is an argument for the "second best."[7]

The first section of this paper addresses the problem of uncertainty, using the state-preference approach to market behavior. It demonstrates that if the returns

[3] For this point of view, see O. Eckstein and S. Marglin.

[4] Hirshleifer (1965 pp. 523–34); (1966, pp. 268–75).

[5] Hirshleifer (1966, pp. 270–75).

[6] Hirshleifer (1966, pp. 270–75).

[7] Hirshleifer (1966, p. 270).

from any particular investment are independent of other components of national income, then the present value of this investment equals the sum of expected returns discounted by a rate appropriate for investments yielding certain returns. This result holds for both private and public investments. Therefore, by adding one plausible assumption to Hirshleifer's formulation, the conclusion can be drawn that the government should behave as an expected-value decision maker and use a discount rate appropriate for investments with certain returns. This conclusion needs to be appropriately modified when one considers the case where there is a corporate income tax.

While this result is of theoretical interest, as a policy recommendation it suffers from a defect common to the conclusions drawn by Hirshleifer. The model of the economy upon which these recommendations are based presupposes the existence of perfect markets for claims contingent on states of the world. Put differently, it is assumed that there are perfect insurance markets through which individuals may individually pool risks. Given such markets, the distribution of risks among individuals will be Pareto optimal. The difficulty is that many of these markets for insurance do not exist, so even if the markets which do exist are perfect, the resulting equilibrium will be sub-optimal. In addition, given the strong evidence that the existing capital markets are not perfect, it is unlikely that the pattern of investment will be Pareto optimal. At the margin, different individuals will have different rates of time and risk preference, depending on their opportunities to borrow or to invest, including their opportunities to insure.

There are two reasons why markets for many types of insurance do not exist. The first is the existence of certain moral hazards.[8] In particular, the fact that someone has insurance may alter his behavior so that the observed outcome is adverse to the insurer. The second is that such markets would require complicated and specialized contracts which are costly. It may be that the cost of insuring in some cases is so high that individuals choose to bear risks rather than pay the transaction costs associated with insurance.

Given the absence of some markets for insurance and the resulting sub-optimal allocation of risks, the question remains: How should the government treat uncertainty in evaluating public investment decisions? The approach taken in this paper is that individual preferences are relevant for public investment decisions, and government decisions should reflect individual valuations of costs and benefits. It is demonstrated in the second section of this paper that when the risks associated with a public investment are publicly borne, the total cost of risk-bearing is insignificant and, therefore, the government should ignore uncertainty in evaluating public investments. Similarly, the choice of the rate of discount should in this case be independent of considerations of risk. This result is obtained not because the government is able to pool investments but because the government distributes the risk associated with any investment among a large number of people. It is the risk-spreading aspect of government investment that is essential to this result.

There remains the problem that private investments may be displaced by public ones yielding a lower return if this rule is followed, although given the absence of insurance markets this will represent a Hicks-Kaldor improvement over the initial situation. Again the question must be

[8] For a discussion of this problem see M. V. Pauly and Arrow (1968).

asked whether the superior position of the government with respect to risk can be made to serve private investors. This leads to a discussion of the government's role as a supplier of insurance, and of Hirshleifer's recommendation that private investment be subsidized in some cases.

Finally, the results obtained above apply to risks actually borne by the government. Many of the risks associated with public investments are borne by private individuals, and in such cases it is appropriate to discount for risk as would these individuals. This problem is discussed in the final section of the paper. In addition, a method of evaluating public investment decisions is developed that calls for different rates of discount applied to different classes of benefits and costs.

I. *Markets for Contingent Claims and Time-Risk Preference*[9]

For simplicity, consider an economy where there is one commodity and there are I individuals, S possible states of the world, and time is divided into Q periods of equal length. Further suppose that each individual acts on the basis of his subjective probability as to the states of nature; let π_{is} denote the subjective probability assigned to state s by individual i. Now suppose that each individual in the absence of trading owns claims for varying amounts of the one commodity at different points in time, given different states of the world. Let $\bar{x}_{isq}$ denote the initial claim to the commodity in period q+1 if state s occurs which is owned by individual i. Suppose further that all trading in these claims takes place at the beginning of the first period, and claims are bought and sold on dated commodity units contingent on a state of the world. All claims can be constructed from basic claims which pay one commodity unit in period q+1, given state s, and nothing in other states or at other times; there will be a corresponding price for this claim, $p_{sq}(s=1, \ldots, S;$ $\mathrm{q}=0, \ldots, \mathrm{Q}-1)$. After the trading, the individual will own claims x_{isq}, which he will exercise when the time comes to provide for his consumption. Let $V_i(x_{i1,0}, \ldots, x_{i1,Q-1}, x_{i2,0}, \ldots, x_{iS,Q-1})$ be the utility of individual i if he receives claims x_{isq} $(s=1, \ldots, S; \mathrm{q}=0, \ldots, \mathrm{Q}-1)$. The standard assumptions are made that V_i is strictly quasi-concave $(i=1, \ldots, I)$.

Therefore each individual will attempt to maximize,

$$V_i(x_{i1,0}, \ldots, x_{i1,Q-1}, x_{i2,0}, \ldots, x_{iS,Q-1}) \tag{1}$$

subject to the constraint

$$\sum_{\mathrm{q}=0}^{\mathrm{Q}-1} \sum_{s=1}^{S} p_{s\mathrm{q}} x_{is\mathrm{q}} = \sum_{\mathrm{q}=0}^{\mathrm{Q}-1} \sum_{s=1}^{S} p_{s\mathrm{q}} \bar{x}_{is\mathrm{q}}$$

Using the von Neumann-Morgenstern theorem and an extension by Hirshleifer,[10] functions U_{is} $(s=1, \ldots, S)$ can be found such that

$$\begin{aligned} V_i(&x_{i1,0}, \ldots, x_{iS,Q-1}) \\ &= \sum_{s=1}^{S} \pi_{is} U_{is}(x_{is0}, x_{is1}, \ldots, x_{iS,Q-1}) \end{aligned} \tag{2}$$

In equation (2) an individual's utility, given any state of the world, is a function of his consumption at each point in time. The subscript s attached to the function U_{is} is in recognition of the fact that the value of a given stream of consumption may depend on the state of the world.

The conditions for equilibrium require that

$$\begin{aligned} &\pi_{is} \frac{\partial U_{is}}{\partial x_{is\mathrm{q}}} = \lambda_i p_{s\mathrm{q}} \qquad (i = 1, \ldots, I; \\ &\qquad s = 1, \ldots, S; \mathrm{q} = 0, \ldots, \mathrm{Q} - 1) \end{aligned} \tag{3}$$

[9] For a basic statement of the state-preference approach, see Arrow (1964) and G. Debreu.

[10] J. von Neumann and O. Morgenstern, and Hirshleifer (1965, pp. 534–36).

where λ_i is a Lagrangian multiplier.

From (3) it follows that

$$(4)\quad \frac{p_{sq}}{p_{rm}} = \frac{\pi_{is}\dfrac{\partial U_{is}}{\partial x_{isq}}}{\pi_{rm}\dfrac{\partial U_{ir}}{\partial x_{irm}}} \qquad (i = 1, \ldots, I;$$

$$r, s = 1, \ldots, S; m, q = 0, \ldots, Q - 1)$$

Insight can be gained by analyzing the meaning of the prices in such an economy. Since trading takes place at time zero, p_{sq} represents the present value of a claim to one commodity unit at time q, given state s. Clearly,

$$\sum_{s=1}^{S} p_{s0} = 1$$

since someone holding one commodity unit at time zero has a claim on one commodity unit, given any state of the world. It follows that p_{sq} is the present value of one commodity at time q, given state s, in terms of a certain claim on one commodity unit at time zero. Therefore, the implicit rate of discount to time zero on returns at time q, given state s, is defined by $p_{sq} = 1/1+r_{sq}$.

Now suppose one considers a certain claim to one commodity unit at time q; clearly, its value is

$$p_q = \sum_{s=1}^{S} p_{sq}$$

and the rate of discount appropriate for a certain return at time q is defined by

$$(5)\quad \frac{1}{1+r_q} = \sum_{s=1}^{S} \frac{1}{1+r_{sq}} = \sum_{s=1}^{S} p_{sq}$$

Given these observations, we can now analyze the appropriate procedure for evaluating government investments where there are perfect markets for claims contingent on states of the world.[11] Consider an investment where the overall effect on market prices can be assumed to be negligible, and suppose the net return from this investment for a given time and state is h_{sq} $(s=1, \ldots, S; q=0, \ldots, Q-1)$. Then the investment should be undertaken if

$$(6)\quad \sum_{q=0}^{Q-1} \sum_{s=1}^{S} h_{sq}p_{sq} > 0,$$

and the sum on the left is an exact expression for the present value of the investment. Expressed differently, the investment should be adopted if

$$(7)\quad \sum_{q=0}^{Q-1} \sum_{s=1}^{S} \frac{h_{sq}}{1+r_{sq}} > 0$$

The payoff in each time-state is discounted by the associated rate of discount. This is the essential result upon which Hirshleifer bases his policy conclusions.[12]

Now suppose that the net returns of the investment were (a) independent of the returns from previous investment, (b) independent of the individual utility functions, and (c) had an objective probability distribution, i.e., one agreed upon by everyone. More specifically, we assume that the set of all possible states of the world can be partitioned into a class of mutually exclusive and collectively exhaustive sets, E_t, indexed by the subscript t such that, for all s in any given E_t, all utility functions U_{is} are the same for any individual i $(i=1, \cdots, I)$, and such that all production conditions are the same. Put differently, for all s in E_t, U_{is} is the same for a given individual, but not necessarily for all individuals. At the same time there is another partition of the states of the world into sets, F_u, such that the return, h_{sq}, is the same for all s in F_u. Finally, we assume that the probability distribution of F_u is independent of E_t and is the same for all individuals.

Let E_{tu} be the set of all states of the world which lie in both E_t and F_u. For any given t and u, all states of the world in

[11] The following argument was sketched in Arrow (1966, pp. 28–30).

[12] Hirshleifer (1965, pp. 523–34).

E_{tu} are indistinguishable for all purposes, so we may regard it as containing a single state. Equations (3) and (5) and the intervening discussion still hold if we then replace s everywhere by tu. However, $U_{is} = U_{itu}$ actually depends only on the subscript, t, and can be written U_{it}. From the assumptions it is obvious and can be proved rigorously that the allocation x_{isq} also depends only on t, i.e., is the same for all states in E_t for any given t, so it may be written x_{itq}. Finally, let π_{it} be the probability of E_t according to individual i, and let π_u be the probability of F_u, assumed the same for all individuals. Then the assumption of statistical independence is written:

$$\pi_{itu} = \pi_{it}\pi_u \tag{8}$$

Then (3) can be written

$$\pi_{it}\pi_u \frac{\partial U_{it}}{\partial x_{itq}} = \lambda_i p_{tuq} \tag{9}$$

Since p_{tuq} and π_u are independent of i, so must be

$$\left(\pi_{it} \frac{\partial U_{it}}{\partial x_{itq}}\right) \Big/ \lambda_i;$$

on the other hand, this expression is also independent of u and so can be written μ_{tq}. Therefore,

$$p_{tuq} = \mu_{tq}\pi_u \tag{10}$$

Since the new investment has the same return for all states s in F_u, the returns can be written h_{uq}. Then the left-hand side of (6) can, with the aid of (10), be written

$$\begin{aligned} &\sum_{Q=0}^{Q-1} \sum_{s=1}^{S} h_{sq} p_{sq} \\ &= \sum_{q=0}^{Q-1} \sum_t \sum_u h_{uq} p_{tuq} \\ &= \sum_{q=0}^{Q-1} \left(\sum_t \mu_{tq} \right) \sum_u \pi_u h_{uq} \end{aligned} \tag{11}$$

But from (10)

$$\begin{aligned} p_q &= \sum_{s=1}^{S} p_{sq} = \sum_t \sum_u p_{tuq} \\ &= \left(\sum_t \mu_{tq} \right) \left(\sum_u \pi_u \right) = \sum_t \mu_{tq}, \end{aligned} \tag{12}$$

since of course the sum of the probabilities of the F_u's must be 1. From (11),

$$\sum_{q=0}^{Q-1} \sum_{s=1}^{S} h_{sq} p_{sq} = \sum_{Q=0}^{Q-1} \frac{1}{1 + r_q} \sum_u \pi_u h_{uq} \tag{13}$$

Equation (13) gives the rather startling result that the present value of any investment which meets the independence and objectivity conditions, equals the expected value of returns in each time period, discounted by the factor appropriate for a certain return at that time. This is true even though individuals may have had different probabilities for the events that governed the returns on earlier investments. It is also interesting to note that each individual will behave in this manner so that there will be no discrepancy between public and private procedures for choosing among investments.

The independence assumption applied to utility functions was required because the functions U_{is} are conditional on the states of the world. This assumption appears reasonable, and in the case where U_{is} is the same for all values of s, it is automatically satisfied. Then the independence condition is simply that the net returns from an investment be independent of the returns from previous investments.

The difficulty that arises if one bases policy conclusions on these results is that some markets do not exist, and individuals do not value assets at the expected value of returns discounted by a factor appropriate for certain returns. It is tempting to argue that while individuals do not behave as expected-value decision makers because of the nonexistence of certain markets for insurance, there is no reason why the government's behavior should not be consistent with the results derived above where the allocation of resources was Pareto optimal. There are two difficulties with this line of argument. First, if we are to measure benefits and costs in terms of individuals' willingness to pay, then we must treat risk in accordance with these individual valuations. Since individuals do not have the opportunities for insuring assumed in the state-preference model, they will not value uncertainty as they would if these markets did exist. Second, the theory of the second best demonstrates that if resources are not allocated in a Pareto optimal manner, the appropriate public policies may not be those consistent with Pareto efficiency in perfect markets. Therefore, some other approach must be found for ascertaining the appropriate government policy toward risk. In particular, such an approach must be valid, given the nonexistence of certain markets for insurance and imperfections in existing markets.

II. *The Public Cost of Risk-Bearing*

The critical question is: What is the cost of uncertainty in terms of costs to individuals? If one adopts the position that costs and benefits should be computed on the basis of individual willingness to pay, consistency demands that the public costs of risk-bearing be computed in this way too. This is the approach taken here.

In the discussion that follows it is assumed that an individual's utility is dependent only upon his consumption and not upon the state of nature in which that consumption takes place. This assumption simplifies the presentation of the major theorem, but it is not essential. Again the expected utility theorem is assumed to hold. The presentation to follow analyzes the cost of risk-bearing by comparing the expected value of returns with the certainty equivalent of these returns. In this way the analysis of time and risk preference can be separated, so we need only consider one time period.

Suppose that the government were to undertake an investment with a certain outcome; then the benefits and costs are measured in terms of willingness to pay for this outcome. If, however, the outcome is uncertain, then the benefits and costs actually realized depend on which outcome in fact occurs. If an individual is risk-averse, he will value the investment with the uncertain outcome at less than the expected value of its net return (benefit minus cost) to him. Therefore, in general the expected value of net benefits overstates willingness to pay by an amount equal to the cost of risk-bearing. It is clear that the social cost of risk-bearing will depend both upon which individuals receive the benefits and pay the costs and upon how large is each individual's share of these benefits and costs.

As a first step, suppose that the government were to undertake an investment and capture all benefits and pay all costs, i.e., the beneficiaries pay to the government an amount equal to the benefits received and the government pays all costs. Individuals who incur costs and those who receive benefits are therefore left indifferent to their pre-investment state. This assumption simply transfers all benefits and costs to the government, and the outcome of the investment will affect government disbursements and receipts. Given that the general taxpayer finances government expenditures, a public investment can be considered an investment in which each individual taxpayer has a very small share.

For precision, suppose that the government undertook an investment and that returns accrue to the government as previously described. In addition, suppose that in a given year the government were to have a balanced budget (or a planned deficit or surplus) and that taxes would be reduced by the amount of the net benefits if the returns are positive, and raised if returns are negative. Therefore, when the government undertakes an investment, each taxpayer has a small share of that investment with the returns being paid through changes in the level of taxes. By undertaking an investment the government adds to each individual's disposable income a random variable which is some fraction of the random variable representing the total net returns. The expected return to all taxpayers as a group equals expected net benefits.

Each taxpayer holds a small share of an asset with a random payoff, and the value of this asset to the individual is less than its expected return, assuming risk aversion. Stated differently, there is a cost of risk-bearing that must be subtracted from the expected return in order to compute the value of the investment to the individual taxpayer. Since each taxpayer will bear some of the cost of the risk associated with the investment, these costs must be summed over all taxpayers in order to arrive at the total cost of risk-bearing associated with a particular investment. These costs must be subtracted from the value of expected net benefits in order to obtain the correct measure for net benefits. The task is to assess these costs.

Suppose, as in the previous section, that there is one commodity, and that each individual's utility in a given year is a function of his income defined in terms of this commodity and is given by $U(Y)$. Further, suppose that U is bounded, continuous, strictly increasing, and differentiable. The assumptions that U is continuous and strictly increasing imply that U has a right and left derivative at every point and this is sufficient to prove the desired results; differentiability is assumed only to simplify presentation. Further suppose that U satisfies the conditions of the expected utility theorem.

Consider, for the moment, the case where all individuals are identical in that they have the same preferences, and their disposable incomes are identically distributed random variables represented by A. Suppose that the government were to undertake an investment with returns represented by B, which are statistically independent of A. Now divide the effect of this investment into two parts: a certain part equal to expected returns and a random part, with mean zero, which incorporates risk. Let $\overline{B}=E[B]$, and define the random variable X by $X=B-\overline{B}$. Clearly, X is independent of A and $E[X]=0$. The effect of this investment is to add an amount $\overline{B}$ to government receipts along with a random component represented by X. The income of each taxpayer will be affected through taxes and it is the level of these taxes that determines the fraction of the investment he effectively holds.

Consider a specific taxpayer and denote his fraction of this investment by s, $0\leqq s\leqq 1$. This individual's disposable income, given the public investment, is equal to $A+sB=A+s\overline{B}+sX$. The addition of sB to his disposable income is valued by the individual at its expected value less the cost of bearing the risk associated with the random component sX. If we suppose that each taxpayer has the same tax rate and that there are n taxpayers, then $s=1/n$, and the value of the investment taken over all individuals is simply $\overline{B}$ minus n times the cost of risk-bearing associated with the random variable $(1/n)X$. The central result of this section of the paper is that this total of the costs of risk-bearing goes to zero as n becomes large. Therefore, for large values of n the value of a public investment almost equals the expected value of that investment.

To demonstrate this, we introduce the function

$$W(s) = E[U(A + s\overline{B} + sX], \qquad 0 \leqq s \leqq 1 \tag{14}$$

In other words, given the random variables A and B representing his individual income before the investment and the income from the investment, respectively, his expected utility is a function of s which represents his share of B. From (14) and the assumption that U' exists, it follows that

$$W'(s) = E[U'(A + s\overline{B} + sX)(\overline{B} + X)] \tag{15}$$

Since X is independent of A, it follows that $U'(A)$ and X are independent; therefore,

$$E[U'(A)X] = E[U'(A)]E[X] = 0$$

so that

$$\begin{aligned} W'(0) &= E[U'(A)(\overline{B} + X)] \\ &= \overline{B}E[U'(A)] \end{aligned} \tag{16}$$

Equation (16) is equivalent to the statement

$$\lim_{s \to 0} \frac{E[U(A + s\overline{B} + sX) - U(A)]}{s} = \overline{B}E[U'(A)] \tag{17}$$

Now let $s = 1/n$, so that equation (17) becomes

$$\lim_{n \to \infty} nE\left[U\left(A + \frac{\overline{B} + X}{n}\right) - U(A)\right] = \overline{B}E[U'(A)] \tag{18}$$

If we assume that an individual whose preferences are represented by U is a risk-averter, then it is easily shown that there exists a unique number, $k(n) > 0$, for each value of n such that

$$E\left[U\left(A + \frac{\overline{B} + X}{n}\right)\right] = E\left[U\left(A + \frac{\overline{B}}{n} - k(n)\right)\right], \tag{19}$$

or, in other words, an individual would be indifferent between paying an amount equal to $k(n)$ and accepting the risk represented by $(1/n)X$. Therefore, $k(n)$ can be said to be the cost of risk-bearing associated with the asset B. It can easily be demonstrated that $\lim_{n\to\infty} k(n) = 0$, i.e., the cost of holding the risky asset goes to zero as the amount of this asset held by the individual goes to zero. It should be noted that the assumption of risk aversion is not essential to the argument but simply one of convenience. If U represented the utility function of a risk preferrer, then all the above statements would hold except $k(n) < 0$, i.e., an individual would be indifferent between being paid $-k(n)$ and accepting the risk $(1/n)X$ (net of the benefit $(1/n)\overline{B}$).

We wish to prove not merely that the risk-premium of the representative individual, $k(n)$, vanishes, but more strongly that the total of the risk-premiums for all individuals, $nk(n)$, approaches zero as n becomes large.

From (18) and (19) it follows that

$$\lim_{n \to \infty} nE\left[U\left(A + \frac{\overline{B}}{n} - k(n)\right) - U(A)\right] = \overline{B}E[U'(A)] \tag{20}$$

In addition, $\overline{B}/n - k(n) \to 0$, when $n \to \infty$. It follows from the definition of a derivative that

$$\lim_{n \to \infty} \frac{E\left[U\left(A + \frac{\overline{B}}{n} - k(n)\right) - U(A)\right]}{\frac{\overline{B}}{n} - k(n)} = E[U'(A)] > 0 \tag{21}$$

Dividing (20) by (21) yields

$$\lim_{n \to \infty} [\overline{B} - nk(n)] = \overline{B} \tag{22}$$

or

$$\lim_{n \to \infty} nk(n) = 0 \tag{23}$$

The argument in (21) implies that $\overline{B}/n - k(n) \neq 0$. Suppose instead the equality held for infinitely many n. Substitution into the left-hand side of (20) shows that $\overline{B}$ must equal zero, so that $k(n) = 0$ for all such n, and hence $nk(n) = 0$ on that sequence, confirming (23).

Equation (23) states that the total of

the costs of risk-bearing goes to zero as the population of taxpayers becomes large. At the same time the monetary value of the investment to each taxpayer, neglecting the cost of risk, is $(1/n)\bar{B}$, and the total, summed over all individuals, is $\bar{B}$, the expected value of net benefits. Therefore, if n is large, the expected value of net benefits closely approximates the correct measure of net benefits defined in terms of willingness to pay for an asset with an uncertain return.

In the preceding analysis, it was assumed that all taxpayers were identical in that they had the same utility function, their incomes were represented by identically distributed variables, and they were subject to the same tax rates. These assumptions greatly simplify the presentation; however, they are not essential to the argument. Different individuals may have different preferences, incomes, and tax rates; and the basic theorem still holds, provided that as n becomes larger the share of the public investment borne by any individual becomes arbitrarily smaller.

The question necessarily arises as to how large n must be to justify proceeding as if the cost of publicly-borne risk is negligible. This question can be given no precise answer; however, there are circumstances under which it appears likely that the cost of risk-bearing will be small. If the size of the share borne by each taxpayer is a negligible component of his income, the cost of risk-bearing associated with holding it will be small. It appears reasonable to assume, under these conditions, that the total cost of risk-bearing is also small. This situation will exist where the investment is small with respect to the total wealth of the taxpayers. In the case of a federally sponsored investment, n is not only large but the investment is generally a very small fraction of national income even though the investment itself may be large in some absolute sense.

The results derived here and in the previous section depend on returns from a given public investment being independent of other components of national income. The government undertakes a wide range of public investments and it appears reasonable to assume that their returns are independent. Clearly, there are some government investments which are interdependent; however, where investments are interrelated they should be evaluated as a package. Even after such groupings are established, there will be a large number of essentially independent projects. It is sometimes argued that the returns from public investments are highly correlated with other components of national income through the business cycle. However, if we assume that stabilization policies are successful, then this difficulty does not arise. It should be noted that in most benefit-cost studies it is assumed that full employment will be maintained so that market prices can be used to measure benefits and costs. Consistency requires that this assumption be retained when considering risk as well. Further, if there is some positive correlation between the returns of an investment and other components of national income, the question remains as to whether this correlation is so high as to invalidate the previous result.

The main result is more general than the specific application to public investments. It has been demonstrated that if an individual or group holds an asset which is statistically independent of other assets, and if there is one or more individuals who do not share ownership, then the existing situation is not Pareto-efficient. By selling some share of the asset to one of the individuals not originally possessing a share, the cost of risk-bearing can be reduced while the expected returns remain unchanged. The reduction in the cost of risk-bearing can then be redistributed to bring about a Pareto improvement. This result is similar to a result derived by Karl Borch. He proved that a condition for Pareto optimality in reinsurance markets requires that every individual hold a share of every independent risk.

When the government undertakes an investment it, in effect, spreads the risk among all taxpayers. Even if one were to accept that the initial distribution of risk was Pareto-efficient, the new distribution of risk will not be efficient as the government does not discriminate among the taxpayers according to their risk preferences. What has been shown is that in the limit the situation where the risk of the investment is spread over all taxpayers is such that there is only a small deviation from optimality with regard to the distribution of that particular risk. The overall distribution of risk may be sub-optimal because of market imperfections and the absence of certain insurance markets. The great advantage of the results of this section is that they are not dependent on the existence of perfect markets for contingent claims.

This leads to an example which runs counter to the policy conclusions generally offered by economists. Suppose that an individual in the private sector of the economy were to undertake a given investment and, calculated on the basis of expected returns, the investment had a rate of return of 10 per cent. Because of the absence of perfect insurance markets, the investor subtracted from the expected return in each period a risk premium and, on the basis of returns adjusted for risk, his rate of return is 5 percent. Now suppose that the government could invest the same amount of money in an investment which, on the basis of expected returns, would yield 6 percent. Since the risk would be spread over all taxpayers, the cost of risk-bearing would be negligible, and the true rate of return would be 6 percent. Further, suppose that if the public investment were adopted it would displace the private investment. The question is: Should the public investment be undertaken? On the basis of the previous analysis, the answer is yes. The private investor is indifferent between the investment with the expected return of 10 percent, and certain rate of return of 5 percent. When the public investment is undertaken, it is equivalent to an investment with a certain rate of return of 6 percent. Therefore, by undertaking the public investment, the government could more than pay the opportunity cost to the private investor of 5 percent associated with the diversion of funds from private investment.

The previous example illustrates Hirshleifer's point that the case for evaluating public investments differently from private ones is an argument for the second best. Clearly, if the advantages of the more efficient distribution of risk could be achieved in connection with the private investment alternative, this would be superior to the public investment. The question then arises as to how the government can provide insurance for private investors and thereby transfer the risks from the private sector to the public at large. The same difficulties arise as before, moral hazards and transaction costs. It may not be possible for the government to provide such insurance, and in such cases second-best solutions are in order. Note that if the government could undertake any investment, then this difficulty would not arise. Perhaps one of the strongest criticisms of a system of freely competitive markets is that the inherent difficulty in establishing certain markets for insurance brings about a sub-optimal allocation of resources. If we consider an investment, as does Hirshleifer, as an exchange of certain present income for uncertain future income, then the misallocation will take the form of under-investment.

Now consider Hirshleifer's recommendation that, in cases such as the one above, a direct subsidy be used to induce more private investment rather than increase public investment. Suppose that a particular private investment were such that the benefits would be a marginal increase in the future supply of an existing commodity, i.e., this investment would neither introduce a new commodity nor affect future prices. Therefore, benefits can be measured at each point in time by the market value of this output, and can be fully captured through the sale of the commodity. Let $\bar{V}$ be the present value of expected net returns, and let V be the present value of net returns adjusted for risk where the certainty rate is used to discount both streams. Further, suppose there were a public investment, where the risks were publicly borne, for which the present value of expected net benefits was P. Since the risk is publicly borne, from the previous discussion it follows that P is the present value of net benefits adjusted for risk. Now suppose that $\bar{V} > P > V$. According to Hirshleifer, we should undertake the private investment rather than the public one, and pay a subsidy if necessary to induce private entrepreneurs to undertake this investment. Clearly, if there is a choice between one investment or the other, given the existing distribution of risk, the public investment is superior. The implication is that if a risky investment in the private sector is displaced by a public investment with a lower expected return but with a higher return when appropriate adjustments are made for risks, this represents a Hicks-Kaldor improvement. This is simply a restatement of the previous point that the government could more than pay the opportunity cost to the private entrepreneur.

Now consider the case for a direct subsidy to increase the level of private investment. One can only argue for direct subsidy of the private investment if $V < 0 < \bar{V}$. The minimum subsidy required is $|V|$. Suppose the taxpayers were to pay this subsidy, which is a transfer of income from the public at large to the private investor, in order to cover the loss from the investment. The net benefits, including the cost of risk-bearing, remain negative because while the subsidy has partially offset the cost of risk-bearing to the individual investor, it has not reduced this cost. Therefore, a direct public subsidy in this case results in a less efficient allocation of resources.

We can summarize as follows: It is implied by Hirshleifer that it is better to undertake an investment with a higher expected return than one with a lower expected return. (See 1965, p. 270.) This proposition is not in general valid, as the distribution of risk-bearing is critical. This statement is true, however, when the costs of risk-bearing associated with both investments are the same. What has been shown is that when risks are publicly borne, the costs of risk-bearing are negligible; therefore, a public investment with an expected return which is less than that of a given private investment may nevertheless be superior to the private alternative. Therefore, the fact that public investments with lower expected return may replace private investment is not necessarily cause for concern. Furthermore, a program of providing direct subsidies to encourage more private investment does not alter the costs of risk-bearing and, therefore, will encourage investments which are inefficient when the costs of risk are considered. The program which produces the desired result is one to insure private investments.

One might raise the question as to whether risk-spreading is not associated with large corporations so that the same result would apply, and it is easily seen that the same reasoning does apply. This can be made more precise by assuming there were n stockholders who were identical in the sense that their utility functions were identical, their incomes were represented by identically distributed random variables, and they had the same share in the company. When the corporation undertakes an investment with a return in a given year represented by B, each stockholder's income is represented by $A+(1/n)B$. This assumes, of course, that a change in earnings was reflected in dividends, and that there were no business taxes. Clearly, this is identical to the situation previously described, and if n is large, the total cost of risk-bearing to the stockholders will be negligible. If the income or wealth of the stockholders were large with respect to the size of the investment, this result would be likely to hold. Note that whether or not the investment is a large one, with respect to the assets of the firm, is not relevant. While an investment may constitute a major part of a firm's assets if each stockholder's share in the firm is a small component of his income, the cost of risk-bearing to him will be very small. It then follows that if managers were acting in the interest of the firm's shareholders, they would essentially ignore risks and choose investments with the highest expected returns.

There are two important reasons why large corporations may behave as risk averters. First, in order to control the firm, some shareholder may hold a large block of stock which is a significant component of his wealth. If this were true, then, from his point of view, the costs of risk-bearing would not be negligible, and the firm should behave as a risk averter. Note in this case that the previous result does not hold because the cost of risk-bearing to each stockholder is not small, even though the number of stockholders is very large. Investment behavior in this case is essentially the same as the case of a single investor.

The second case is when, even though from the stockholder's point of view, risk should be ignored, it may not be in the interest of the corporate managers to neglect risk. Their careers and income are intimately related to the firm's performance. From their point of view, variations in the outcome of some corporate action impose very real costs. In this case, given a degree of autonomy, the corporate managers, in considering prospective investments, may discount for risk when it is not in the interest of the stockholders to do so.

Suppose that this were the case and also suppose that the marginal rate of time preference for each individual in the economy was 5 percent. From the point of view of the stockholders, risk can be ignored and any investment with an expected return which is greater than 5 percent should be undertaken. However, suppose that corporate managers discount for risk so that only investments with expected rates of return that exceed 10 percent are undertaken. From the point of view of the stockholders, the rate of return on these investments, taking risk into account, is over 10 percent. Given a marginal rate of time preference of 5 percent, it follows that from the point of view of the individual stockholder there is too little investment. Now suppose further that the government were considering an investment with an expected rate of return of 6 percent. Since the cost of risk-bearing is negligible, this investment should be undertaken since the marginal rate of time preference is less than 6 percent. However, in this case, if the financing were such that a private investment with a 10 percent expected rate of return is displaced by the public investment, there is a loss because in both cases the risk is distributed so as to make the total cost of risk-bearing negligible. The public investment should be undertaken, but only at the expense of consumption.

III. *The Actual Allocation of Risk*

In the idealized public investment considered in the last section, all benefits and costs accrued to the government and were distributed among the taxpayers. In this sense, all uncertainty was borne collectively. Suppose instead that some benefits and costs of sizeable magnitudes accrued directly to individuals so that these individuals incurred the attendant costs of risk-bearing. In this case it is appropriate to discount for the risk, as would these individuals. Such a situation would arise in the case of a government irrigation project where the benefits accrued to farmers as increased income. The changes in farm income would be uncertain and, therefore, should be valued at more or less than their expected value, depending on the states in which they occur. If these increases were independent of other components of farm income, and if we assume that the farmer's utility were only a function of his income and not the state in which he receives that income, then he would value the investment project at less than the expected increase in his income, provided he is risk averse. If, however, the irrigation project paid out in periods of drought so that total farm income was not only increased but also stabilized, then the farmers would value the project at more than the expected increase in their incomes.

In general, some benefits and costs will accrue to the government and the uncertainties involved will be publicly borne; other benefits and costs will accrue to individuals and the attendant uncertainties will be borne privately. In the first case the cost of risk-bearing will be negligible; in the second case these costs may be significant. Therefore, in calculating the present value of returns from a public investment a distinction must be made between private and public benefits and costs. The present value of public benefits and costs should be evaluated by estimating the expected net benefits in each period and discounting them, using a discount factor appropriate for investments with certain returns. On the other hand, private benefits and costs must be discounted with respect to both time and risk in accordance with the preferences of the individuals to whom they accrue.

From the foregoing discussion it follows that different streams of benefits and costs should be treated in different ways with respect to uncertainty. One way to do this is to discount these streams of returns at different rates of discount ranging from the certainty rate for benefits and costs accruing to the government and using higher rates that reflect discounting for risk for returns accruing directly to individuals. Such a procedure raises some difficulties of identification, but this problem does not appear to be insurmountable. In general, costs are paid by the government, which receives some revenue, and the net stream should be discounted at a rate appropriate for certain returns. Benefits accruing directly to individuals should be discounted according to individual time and risk preferences. As a practical matter, Hirshleifer's suggestion of finding the marginal rate of return on assets with similar payoffs in the private sector, and using this as the rate of discount, appears reasonable for discounting those benefits and costs which accrue privately.

One problem arises with this latter procedure which has received little attention. In considering public investments, benefits and costs are aggregated and the discussion of uncertainty is carried out in terms of these aggregates. This obscures many of the uncertainties because benefits and costs do not in general accrue to the same individuals, and the attendant uncertainties should not be netted out when considering the totals. To make this clear, consider an investment where the benefits and costs varied greatly, depending on the state of nature, but where the difference between total benefits and total costs was constant for every state. Further, suppose that the benefits and costs accrued to different groups. While the investment is certain from a social point of view, there is considerable risk from a private point of view.

In the case of perfect markets for contingent claims, each individual will discount the stream of costs and benefits accruing to him at the appropriate rate for each time and state. However, suppose that such markets do not exist. Then risk-averse individuals will value the net benefits accruing to them at less than their expected value. Therefore, if net benefits accruing to this individual are positive, this requires discounting expected returns at a higher rate than that appropriate for certain returns. On the other hand, if net benefits to an individual are negative, this requires discounting expected returns at a rate lower than the certainty rate. Raising the rate of discount only reduces the present value of net benefits when they are positive. Therefore, the distinction must be made not only between benefits and costs which accrue to the public and those which accrue directly to individuals, but also between individuals whose net benefits are negative and those whose benefits are positive. If all benefits and costs accrued privately, and different individuals received the benefits and paid the costs, the appropriate procedure would be to discount the stream of expected benefits at a rate higher than the certainty rate, and costs at a rate lower than the certainty rate. This would hold even if the social totals were certain.

Fortunately, as a practical matter this may not be of great importance as most costs are borne publicly and, therefore, should be discounted using the certainty rate. Benefits often accrue to individuals, and where there are attendant uncertainties it is appropriate to discount the expected value of these benefits at higher rates, depending on the nature of the uncertainty and time-risk preferences of the individuals who receive these benefits. It is somewhat ironic that the practical implication of this analysis is that for the typical case where costs are borne publicly and benefits accrue privately, this procedure will qualify fewer projects than the procedure of using a higher rate to discount both benefits and costs.

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