Model Approximation via Dimension Reduction

Ulf Dieckmann & Colin P. Williams

Dynamics of Computation Group,
Xerox Palo Alto Research Center, Palo Alto, CA 94304, U.S.A.
dieckman@parc.xerox.com & cwilliam@parc.xerox.com

Abstract

In the initial stages of refining a mathematical model of a real-world dynamical system, one is often confronted with many more variables and coupled differential equations than one intuitively feels should be sufficient to describe the system. Yet none of the variables may seem so irrelevant as to be excludable nor so dominant as to explain the overall dynamics. Part of the problem might even be that one has been forced to formulate the problem in some "convenient" but not necessarily "ideal" set of variables. In such a circumstance one wishes to simplify the model by an approximation.

In this paper we present a numerical technique called *Extended Adiabatic Elimination* (so named because it generalizes *Direct Adiabatic Elimination*) for automatically approximating a dynamical model by an equivalent one involving fewer, more appropriate, variables. Given, a set of coupled ordinary differential equations and a spatial domain of interest, EAE first tests whether the model is approximable and if so, returns the approximate model whose independent variables are called *order parameters*, together with an indication of the temporal domain of validity of the approximation. The order parameters are composite variables, built from those in the original model, and represent, in a sense, an "ideal" set of variables for the given problem. We explain the theoretical basis underpinning EAE and describe the steps in the procedure with respect to a running example. EAE is both more accurate than DAE and is capable of tackling phase spaces of dimensions beyond those that DAE can handle. Currently, the automated parts of the system can deal with low dimensional phase spaces but, in principle, the algorithm appears to be readily generalisable.

Keywords: approximation, phase space, numerical, adiabatic elimination

1 Introduction

In the initial stages of mathematically modelling some dynamical system one may be faced with a rather daunting set of equations. In order to make such a model manageable, one typically tries to simplify it using various model approximation techniques such as piecewise linearisation [9, 10, 11 ,12], order of magnitude reasoning [6, 7], differential qualitative analysis [14, 15], exaggeration [15], timescale abstraction [4], analytic abduction [16, 17] or qualitative pertubation analysis [5]. Unfortunately, it can be very difficult to assess, a priori, the relative importance of the variables appearing in the model, whether a given approximation strategy preserves the key features of the dynamics and over what scale the approximation is valid.

In this paper, we present a new, computationally tractable, procedure, called *Extended Adiabatic Elimination* (EAE), for automatically analysing a system of coupled first order differential equations, that is able to provide answers to these questions. This procedure grew out of an established analytic technique, called *Direct Adiabatic Elimination* (DAE), and is superior to it in several respects. So, although our technique is new, it inherits a respectable pedigree.

Extended adiabatic elimination can be applied to dynamical systems in which the timescales characterizing the dynamics of its components are not all of the same order of magnitude. Thus from the initial set of variables, one can compose new variables describing only the dynamics on the large timescales. Then because the "fast" components relax quickly, the entire dynamics can be characterized rather well by evolution equations along just the "slow" components. This partitioning, together with a test of whether the model is approximable, allows us:

1. to reduce the dimension of the original model,
2. to estimate, from a knowledge of the rate at which the "fast" variables relax, a natural timescale after which the approximation will be valid and
3. to identify the "slow" variables (also called *order parameters*) as the key descriptors of the dynamics.

The EAE procedure can therefore compute information which is much harder to extract from other model approximation techniques.

In the following section we summarize those features of dynamical systems theory needed to understand our technique and describe a family of approximation

procedures all of which reduce dimension in some manner. Next we outline the extended adiabatic elimination procedure, explaining its inputs, outputs and essential mechanisms. This is followed in section 3 by an instantiation of the steps in the EAE procedure in the context of a simple, but non-trivial, example. Finally, we discuss the relative advantages of EAE in comparison to other approximation techniques and mention some open issues.

2 Model Approximation via Dimension Reduction

In this section we develop the basic concepts of an approach to formalize the process of model approximation. After introducing basic notions from dynamical systems theory we investigate a variety of dimension reducing operations commonly applied in scientific modelling. We then give the specifications for a general dimension reduction procedure — enabling an automatisation of these operations.

Phase Space Dynamics

A *dynamical system* may be represented in terms of a set of first order ordinary differential equations

$$\frac{d}{dt}\underline{x} = F\underline{x}$$

where $\underline{x}(t)$ is the instantaneous *state* of the dynamical system at time t, denoting a vector of n variables $x_1(t), \ldots, x_n(t)$, the so-called *state variables*. The space formed by the Cartesian product of these state variables is called the *phase space* and the instantaneous state of the system may therefore be represented by a single representative point in this phase space. In particular, one has the initial state $\underline{x}(0)$, which is just the state at time $t = 0$.

As the dynamical system evolves from some initial state, determined by the right hand side $F\underline{x}$ of the differential equations, often referred to as the *force*, the set of states which $\underline{x}(t)$ traces out with $0 \leq t < \infty$ is called a *trajectory*. In general, this evolution is described by the *propagator* P_t which maps an initial state $\underline{x}(0)$, at time $t = 0$, onto the state $\underline{x}(t)$ at time t i.e.

$$\underline{x}(t) = P_t\ \underline{x}(0)$$

The propagator and the force are related via $P_t = \exp(Ft)$.

Trajectories consisting of only one state in phase space are called *fixed points*. Certain types of fixed point (stable fixed points) can be the asymptotic state a dynamical system. More generally, a wider class of asymptotic behaviour can be characterized by considering properties of trajectories. Let A be a manifold in the phase space with dimension smaller than n and let D a subset of the phase space. If the distance between A and all trajectories with initial states in D tends to 0 as $t \to \infty$, A will be called an *attractor* and the largest subset D its *domain*.

A more complete introduction into dynamical system theory may be found in [8, 13]. Applications on reasoning about dynamical models are given in [9, 10, 11, 12] and [18, 19, 20].

Ways of Reducing Dimension

Next we turn to the question how phase space analysis can contribute to the approximation of a model represented in the canonical form mentioned above.

There are a number of procedures yielding a simplified or reduced version of a dynamical model:

1. *spatial graining*
   A dynamical variable with a *small spatial variance* can be replaced by a constant.
2. *temporal graining*
   A spatially *bounded but fast changing* variable may be replaced by a constant equal to its temporal average.
3. *constrained elimination*
   If the dynamical system roughly obeys a *constraint* (sometimes called an *equation of state* or a *conservation law*) it is possible to remove as many variables as there are constraints.
4. *asymptotic elimination*
   The dynamics of a system possessing an attractor can be approximated by considering the asymptotic temporal limit which regards the motion as being *on* the attractor.
5. *adiabatic elimination*
   If the dynamical system contains variables with relaxation times of different magnitudes — so called *fast and slow variables* — elimination of the former will simplify the model. This paper focuses on this kind of approximation.

What are the similarities and differences between these distinct notions of approximation?

The graining techniques require the user to specify a spatial and temporal *resolution* for the approximation. In contrast, the elimination techniques, in addition to requiring the spatio-temporal resolution, also need to be given some *domain of interest* delimited by upper and lower bounds for the spatial and temporal variables. These specifications may be given by $(\underline{x}_{min}, \underline{x}_{max}, \underline{x}_{res})$ and $(t_{min}, t_{max}, t_{res})$.

In the cases of spatial and temporal graining the explicit definition of $\underline{x}_{res}$ and t_{res} allows us to neglect dynamical degrees of freedom. In case of the asymptotic elimination, one can intuitively see that the relaxation time towards the attractor defines a natural lower bound for the temporal domain t_{min}. The upper bound t_{max} is usually much larger than t_{min}. Similarly, with adiabatic elimination, a natural lower bound for t_{min} is given by the relaxation time of the fast variables. Constrained elimination can be interpreted as a limiting case with regard to adiabatic and asymptotic elimination with $t_{min} = 0$. The significance of $\underline{x}_{min}$ and $\underline{x}_{max}$ is based on the fact that the phase space can be partitioned into several domains of attraction. Hence the choice of the spatial domain of interest dictates which domain of attraction will be selected.

Another property of the above approximation techniques is their *effect on the dimension* of the dynamical system. The dimension can be defined by the number of first-order differential equations describing the model. In the cases of spatial and temporal graining the dimension is reduced by one due to the elimination of one variable. In the constrained elimination the dimension is reduced by the number of conservation laws, in the asymptotic elimination by the number of relaxing variables and in the adiabatic elimination by the number of fast variables.

A final point to note concerns the importance of choosing the right coordinate system in which to make the approximations. To do so, it is important to distinguish between the *variables* and the *components* of a dynamical process. A simple example will help to illustrate this idea. Imagine a particle moving on the periphery of a circle embedded in a plane — not accurate but with a small wobble. Thus the trajectory of the particle will be close to the one-dimensional circumference and can be essentially made one-dimensional by spatial graining. Nevertheless, with respect to Cartesian coordinates the variations of none of the variables x and y can be neglected although we intuitively tend to characterize

this motion as possessing one component. It is evident that we have to apply polar coordinates in order to declare the variation of the r component as negligible and confine the dynamics to the ϕ component. Only in that coordinate system the notions of variables and components are equivalent. This indicates that it is in general necessary to *transform the variables* of a dynamical system to a suitable set of coordinates before eliminating degrees of freedom. Finding a suitable coordinate system is therefore an important issue.

In this paper we build upon the technique of adiabatic elimination. We restrict our concern to this case for two reasons:

1. Adiabatic elimination is the most general of the three elimination techniques, and in fact, can be regarded as subsuming all the others.
2. Spatial and temporal graining require an explicit time and space resolution to be *given*. By contrast, adiabatic elimination, although requiring a spatial domain and resolution to be supplied, can *compute* the temporal bound above which the approximate model is guaranteed to be valid. Thus this procedure can self-diagnose its temporal domain of validity.

The Dimension Reduction Procedure

Next we give an outline of a general procedure which is capable of approximating a model by reducing its dimension. In the following section, we will then propose a set of specific means to execute the individual steps of this general procedure, illustrated by means of a running example.

Given a system which possesses relaxing components (an assumption typically fulfilled by natural systems due to the dissipative effects e.g. of friction) and given the fact that its relaxation times are not all of the same order of magnitude, i.e. that it comprises slow and fast components (which is again quite common in nature, cf. [4]), we claim that the dimension of the model can be reduced by the number of fast components, denoted by n_f. The reason for this reduction is the following: After a characteristic time t_{min} all the fast components will be practically relaxed. This imposes a set of n_f constraints on the equations of the dynamical system whose dynamics will therefore be confined to an $n - n_f$ dimensional manifold in phase space. We will call such a manifold a *transient attractor*. The dynamics can thus be described *on* this transient attractor instead of the entire phase space. This is the essential step in the dimension reduction procedure.

As already pointed out it is normally necessary to transform the variables of the dynamical system first to a coordinate system which is more appropriate for the elimination of variables. We now see that this new coordinate system for the phase space has to be chosen so that it includes a coordinate system for the transient attractor. Then by neglecting all variables of this new system which are necessary to refer to points not on the transient attractor we get the desired reduction of dimension. In addition to that we have to know the description of the dynamics on the transient attractor, i.e. the propagator of the reduced system. Finally we should be able to interpret a state on the transient attractor in terms of the variables of the entire phase space, which means we have to apply the inverse of the above coordinate transformation.

We summarize the previous discussion in the following definition and procedure:

Definition: In this paper we will regard a model as *approximable* if after a characteristic time t_{min} the dynamics is essentially confined to a manifold in phase space with a dimension smaller than that of phase space.

Figure 1 presents an outline of the *EAE procedure* (EAE for *Extended Adiabatic Elimination*). The individual steps are explained in detail in section 3 (note that the order of the steps is necessary in order to automate the procedure). A graphical illustration of EAE is given in figure 2.

The Extended Adiabatic Elimination Procedure

Input: The definition of a model, given as $\frac{d}{dt}\underline{x} = F\underline{x}$, together with the spatial domain of interest.
Output: The approximate model, including the reduced dimension, and the temporal domain of validity.

A. *Is the model approximable?*
Decide, if the given model is approximable according to the definition. In this case proceed.

B. *The transient attractor*
Obtain the manifold in phase space corresponding to the complete relaxation of all fast variables, i.e. the transient attractor, and derive a proper coordinate system on that transient attractor.

C. *The coordinate transformation*
Construct the mapping C which interpretes states on the transient attractor in terms of the coordinates of phase space.

D. *The reduced propagator*
Construct the propagator of the reduced system on the transient attractor, denoted by Π_t.

E. *The projector*
Construct the mapping R which projects the phase space onto the transient attractor. This dimension reducing projection R is defined so that trajectories of initial states in phase space converge to trajectories of their mapping.

Figure 1

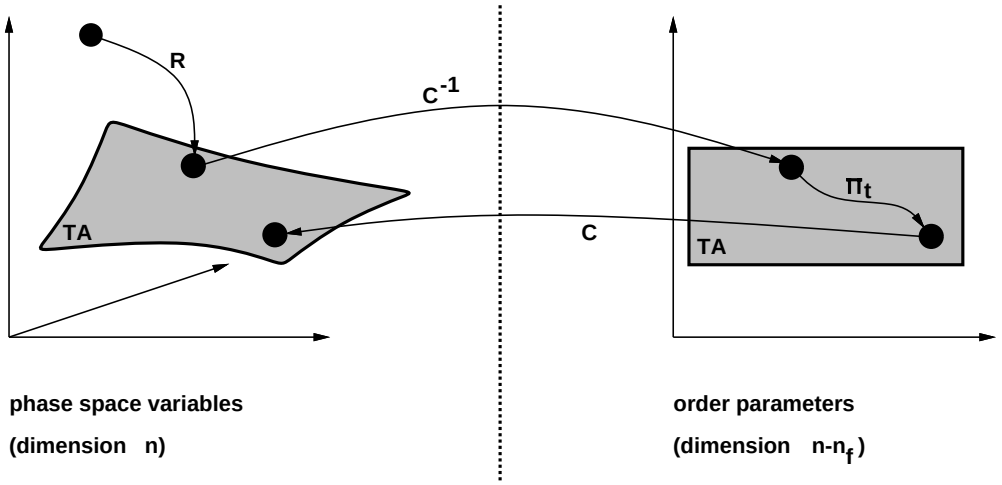

Figure 2

An illustration of EAE. The transient attractor is denoted by TA, states are indicated by points, the arrows show the effect of the operators.

By applying the dimension reduction procedure EAE we get some significant insight with regard to the original model:

1. outputs of step A.
   We learn about the number of fast components in the system and their relaxation times enabling us to predict t_{min} and thereby the time interval (t_{min}, ∞) for which the approximation will be valid.
2. outputs of step B.
   We find a set of new coordinates corresponding to the dynamics of the slow components on the transient attractor — we will denote them by $\underline{\xi}$. Because of the coordinate transformation these coordinates $\underline{\xi}$ will be composite variables with regard to the original variables $\underline{x}$. They identify the relevant or significant aspects of the dynamics and thus can contain important information about the system. Due to a tradition in physics we call those composite variables $\underline{\xi}$ *order parameters*.
3. outputs of step C., D. and E.
   Most important we obtain an approximation for the dynamics in terms of these order parameters. It is evident that the dynamical description $\underline{x}(t) = C\Pi_t C^{-1} R\underline{x}(0)$ is valid for initial states on the transient

attractor. For other states this is an approximation. We can express this fact more precisely by an operator equation:

$$P_t \approx C\Pi_t C^{-1} R$$

It is exactly in this sense we call the reduced model an approximation of the original model.

The promise of this approach is to describe the dynamics of a high dimensional model by reducing it to an approximate lower dimensional model, propagating it in this reduced description and then expanding its dimension again.

3 Extended Adiabatic Elimination

In this section we present a detailed example, illustrating the steps of the extended adiabatic elimination procedure. After introducing a model for this example each of the following divisions mirrors a step in the outline of the EAE procedure.

The Example Model

Consider an object moving in the following potential V:

$$V(x,y) = x^4 + y^4 + 2x^2y^2 - 4x^2 - 4y^2 + 2y$$

The phase space of this system is the $xy-$plane. The dynamics of the object should correspond to the motion of an overdamped particle in the given potential (e.g. damped by a medium with very high viscosity). The canonical form of this system is thus provided by $\frac{d}{dt}(x,y) = -\nabla V(x,y)$ with $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ which yields explicitly:

$$\begin{aligned}\frac{d}{dt}x &= -4x^3 - 4xy^2 + 8x \\ \frac{d}{dt}y &= -4y^3 - 4x^2y + 8y - 2\end{aligned}$$

We choose a two dimensional system to facilitate graphical representation. The spatial domain of interest should be given as $(x,y) \in \{[-2,+2] \times [-2,+2]\}$. An illustration of the potential is provided in figure 3.

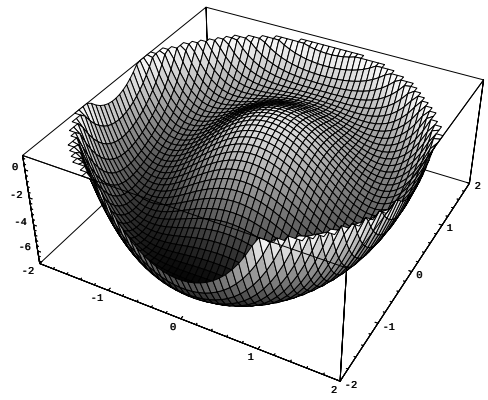

Figure 3

The potential $V(x, y)$ of the example model. The phase space variables x and y are plotted to the left resp. to the right. Note that we chose the differential equations of the model as to correspond to the physical motion of an overdamped particle in this potential (e.g. damped by a medium with very high viscosity).

A. Is the Given Model Approximable?

We will first address the question of approximability. According to the definition given in the previous section we have to provide a means to decide if "after a characteristic time t_{min} the dynamics is essentially confined to a manifold in phase space with a dimension smaller than that of phase space". To do so we need a dimension measure.

In a homogeneous object the dimension d describes the variation of a volume V with its length scale R. Consider e.g. the dependence of a spherical volume on its radius $V(R) = \frac{4}{3}\pi R^3$ or the relation between the "volume" of a circle, which is the enclosed area, and its radius $V(R) = \pi R^2$. For general dimensions V and R obey a power law according to $V(R) \propto R^d$. In practice objects can only be sampled by a finite number of discrete points. What is the volume of a point set? There are several ways to define this notion (cf. [1]), one very natural approach is the following. Consider the quantity C defined as:

$$C(R) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \Theta\big(R - \big|\underline{x}_i - \underline{x}_j\big|\big)$$

$\{\underline{x}_i\}$ is a set of N points. Θ denotes the Heaviside function with $\Theta(x \geq 0) = 1$ and $\Theta(x < 0) = 0$. $|\cdot|$ is the Euclidian length of a vector. The Function Θ simply will yield 1 iff the point $\underline{x}_j$ is within a radius R of the point $\underline{x}_i$ so that the sum $\frac{1}{N}\Sigma_j$ gives the fraction of points within the radius R of $\underline{x}_i$. The sum $\frac{1}{N}\Sigma_i$ averages this fraction over all points $\underline{x}_i$ of the entire set — so $C(R)$ denotes the average fraction of points within R. The so defined quantity C has the desired property of being a measure for the volume of a point set. This implies a power law $C(R) \propto R^d$ between C and R. The dimension d thus is defined by the straight line portion in a plot of $\log C$ vs. $\log R$.

So, in order to answer whether the model is approximable, we have to calculate the dimension of a point set sampling the relevant part of the model's phase space and evolving according to the model's propagator. This will show how the dimension of the sampling set is changing with time. If the initial dimension of the point set (equal to n, the dimension of phase space) is decreasing fast by the number n_f this will indicate not only the existence of n_f fast variables in that model but will also provide the characteristic time t_{min} of their relaxation.

Applying this operation to our example model yields the d vs. t plot in figure 5. To understand the significance of this plot it is helpful to compare it to figure 4 showing snapshots of the actual dynamics of the sample set. Note that the drop from $d = 1$ to $d = 0$ is due to the existence of a finite spatial resolution. The results of this step are summarized by $n_f = 1$ and $t_{min} \approx 1$.

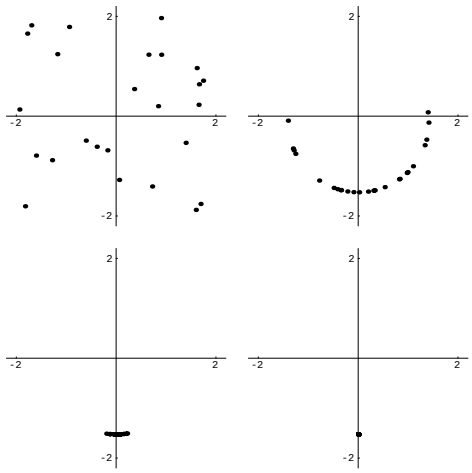

Figure 4

Snapshots of the dynamics of a point set sampling the spatial domain of interest. The diagrams show the point set at times $t = 0, 1, 3, 8$ (from left to right and from upper row to lower, x horizontal and y vertical).

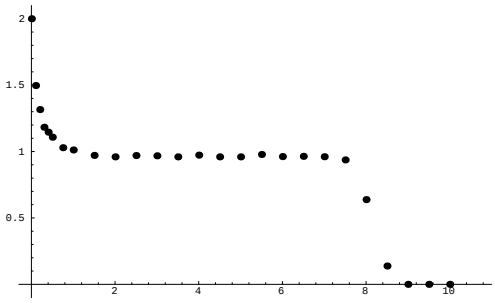

Figure 5

Evolution of the point set's dimension (vertical) with time (horizontal). Note the sharp drop from dimension $d = 2$ to $d = 1$ within $0 < t < 1$ due to the relaxation of the fast components. The time of the drop from $d = 1$ to $d = 0$ is dependent on the choice of the spatial resolution (here 10^{-4}).

B. The Transient Attractor

The most important step in the EAE procedure is to obtain the transient attractor of the model. The analysis performed in step A. contains all the information needed to proceed.

In particular by knowing n_f and t_{min} we now can conclude that after the time t_{min} the dynamics will be confined to a $n - n_f$ dimensional manifold embedded in phase space. So we already have a characterization of the transient attractor: it is sampled by the union of those point sets obtained in step A. which correspond to $t > t_{min}$. If necessary, interpolation between these points can be applied to obtain extra points of the transient attractor. We can cover the transient attractor by a mesh of points, thereby describing it to the desired accuracy.

The issue of choosing a coordinate system on the transient attractor is very simple in the case $n - n_f = 1$. A coordinate system for a one dimensional manifold is given by arbitrarily picking one point on that manifold, called the origin, and

defining the coordinate of a point on the manifold as the directed length of a curve along the manifold connecting the point with the origin. To construct a coordinate system in the case of a higher dimensional transient attractor one has to consider the topological properties of that manifold (a feature not yet automated in the implementation of EAE). The so constructed set of coordinates on the transient attractor is then identical to the set of order parameters $\underline{\xi}$.

The transient attractor of our model turns out to be a circle shaped line, see figure 6. This is in agreement with the intuition that the valley floor of the potential, compare figure 3, and the transient attractor should coincidence. We chose (arbitrarily) the origin of the coordinate system as the point on the transient attractor with the largest value of the potential.

C. The Coordinate Transformation

After we constructed a coordinate system on the transient attractor we now have to find the map C casting the order parameters $\underline{\xi}$ onto the original variables $\underline{x}$. Being a mere coordinate transformation this is the easiest step in our procedure. The necessary information is provided by the shape of the transient attractor in phase space. We get the map C by applying the definition of the variables $\underline{\xi}$ in order to obtain locations in phase space (on the transient attractor) which can be interpreted in terms of the original variables $\underline{x}$.

The map C for our model is presented in figure 7.

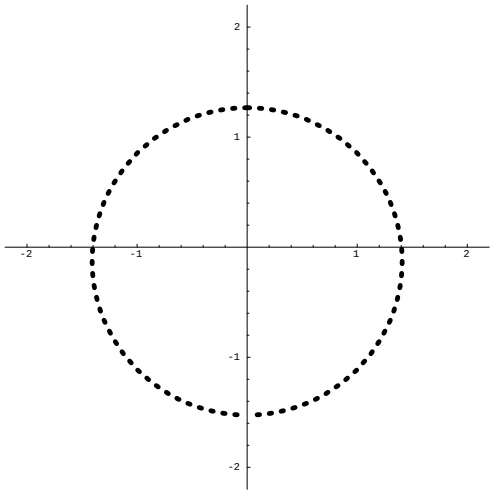

Figure 6

The transient attractor of the example model. Note its coincidence with the valley floor of the potential in figure 3.

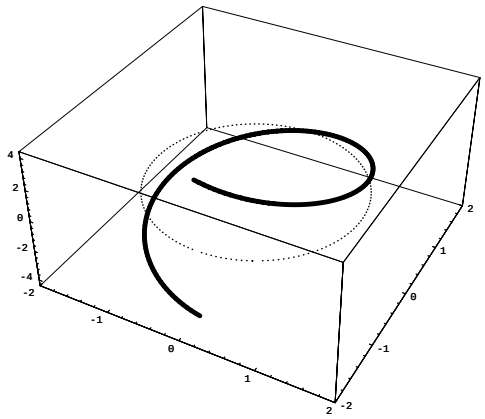

Figure 7

The coordinate transformation C of the example model (ξ vertical, x to the left, y to the right). It is only defined on the transient attractor, which is shown (dashed) on the plane $\xi = 0$.

D. The Reduced Propagator

The next task is to derive the propagator Π_t of the reduced dynamical system formed by the order parameters $\underline{\xi}$. For that purpose we employ the map C constructed in the last step. The inverse C^{-1} of this coordinate transformation is only uniquely defined on the transient attractor. By this reversal we cast the original variables $\underline{x}$ onto the order parameters $\underline{\xi}$, i.e. $\underline{\xi} = C^{-1}\underline{x}$. Exactly the same transformation has to be applied in order to cast the force on the original variables onto the force on the order parameters. We denote the latter by $\Phi\underline{\xi}$ and conclude $\Phi\underline{\xi} = C^{-1}F\underline{x}$. Because force and propagator of a dynamical system are equivalent this is already all we have to know for the construction of Π_t.

Figure 8 illustrates the dependence of the force $\Phi\xi$ on the order parameter ξ in the example model. Note that according to the definition of the coordinate system in step B. ξ denotes the arc length on the one dimensional transient attractor shown in figure 6.

E. The Projector

In the final step of this procedure we construct the dimension reducing map R. It projects a state of the phase space onto the transient attractor. The projection operation has to be done in a manner that the state and its projection converge due to the relaxation of the fast variables. For the purpose of this construction we simply utilize the evolution of point sets already obtained in step A. First we follow the propagation of an initial state until after a time τ its distance to the transient attractor becomes smaller than the spatial resolution. Then we transform the coordinates of that state on the transient attractor from the original variables to the order parameters by means of C^{-1}. Next we trace this state backward on the transient attractor by applying the inverse Π_τ^{-1} of the reduced propagator. Finally we interpret the back propagated state in terms of the original variables by means of C. The so defined state is the mapping of the initial state under R. Note that we can interpolate between initial conditions in order to increase the accuracy of the mapping. Moreover, the time differences between the snapshots derived in step A. need not to be small — the construction of the map, R, tends to be very stable.

In figure 9 we provide a visualization of the projector R for our example model.

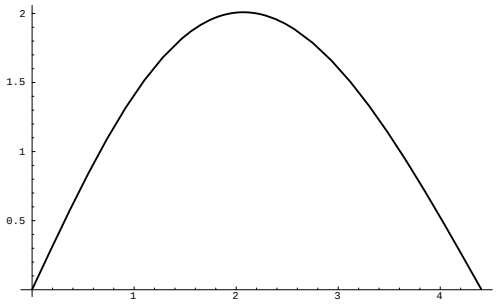

Figure 8

The reduced propagator Π_t of the example model is given by the force $\Phi\xi$. The diagram shows the dependence of $\Phi\xi$ (vertical) on the order parameter ξ (horizontal).

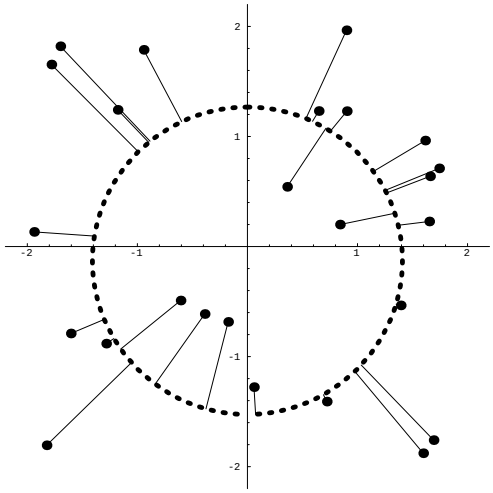

Figure 9

The projector of our example model maps the two phase space coordinates onto the transient attractor. Four dimensions would be necessary for a complete diagram. Therefore we just show the mapping of a few states (the initial point set of figure 4).

4 Discussion

In this paper we investigated basic techniques of model approximation extensively applied by scientists. We pointed out that an essential feature of these techniques is the reduction of dimension and gave a detailed analysis of how to formalize adiabatic elimination, the most promising method. By providing the general dimension reduction procedure EAE for dynamical systems represented by a set of first order differential equations we simplified the whole operation of adiabatic elimination to the construction of three maps — the coordinate transformation, the reduced propagator and the projector — thereby opening the opportunity for automatization.

We compare our *extended adiabatic elimination* (EAE) to the analytic technique, known as *direct adiabatic elimination* (DAE), cf. [2, 3, 13]. Of course symbolic reasoning by means of analytic algorithms is superior to the numeric approach reported in this paper. But unfortunately the range of DAE is very restricted. It will only be applicable to dynamical systems if these possess fixed

points and if the curl of their force vanishes at such a point. Furthermore DAE uses eigensystem analysis, which involves the solution of a characteristic polynomial whose degree increases with the dimension of phase space. Thus DAE can be conducted analytically only for systems with dimensions up to four. EAE is not only more general than DAE, it even proves to outperform its analytic counterpart. In figure 10 we contrast instances for the exact solution of the example model both with the approximations obtained by direct adiabatic elimination and our procedure. Direct adiabatic elimination fails to capture essential qualitative features of the dynamics.

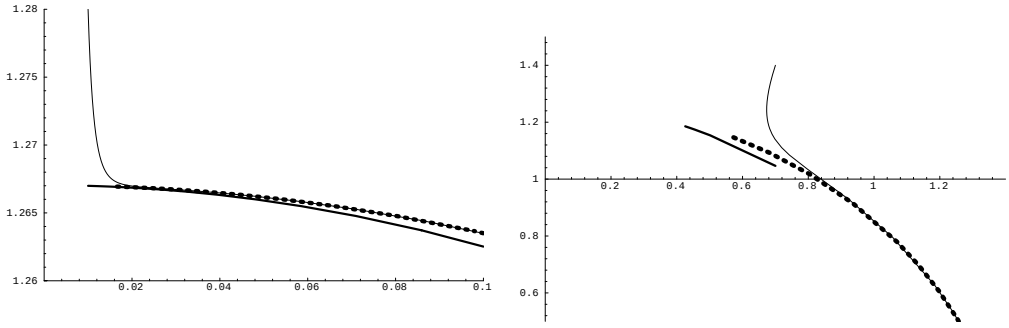

Figure 10

These diagrams show a comparison of two approximation techniques. Direct adiabatic elimination DAE (thick solid) is contasted to our approach of extended adiabatic elimination EAE (thick dashed) with respect to the exact solution of the model (thin solid). The plotted lines are trajectories in the phase space of x (horizontal) and y (vertical). In the case presented to the left DAE and EAE are of comparable quality although DAE is less accurate. On the right we give an example where the standard DAE technique grossly mispredicts the dynamics whereas our EAE procedure quickly converges to the exact solution.

The EAE procedure presented in this paper can easily be generalized. It can be applied to higher dimensional phase spaces and other asymptotic attractors (e.g. limit cycles instead of fixed points). In section 2 we claimed the procedure would be able to handle constrained and asymptotic elimination as well. This is due to the fact that in these cases the basic steps would remain the same, except

for replacing the notion of the transient attractor by manifolds representing either the constraints or the vanishing of all relaxing variables.

Apart from the construction of the coordinate system for higher dimensional transient attractors the entire extended adiabatic elimination procedure EAE is implemented for one dimensional transient attractors. We are working on generalizing it to higher dimensional phase spaces.

5 Summary & Conclusions

The fundamental idea underlying the extended adiabatic elimination procedure is that for many real-world dynamical systems, the timescale characterizing certain components of the dynamics is much larger than that of others, inducing a natural partition of the variables into “fast” and “slow” ones. Both fast and slow components are ultimately evolving towards some asymptotic attractor but the fast ones relax much quicker. This means that after some characteristic time the overall dynamics can be well described by just the evolution of the slow components.

This kind of approximation, which grew out of well established analytic techniques, is most suited to the initial stages of refining a mathematical model. At this stage there often appear to be more variables and coupled differential equations than one intuitively feels should be sufficient to describe the system. Yet none of the variables may seem so irrelevant as to be excludable nor so dominant as to explain the overall dynamics. EAE is an accurate substitute for that intuition and allows one to strip away the irrelevant features to obtain a lower dimensional model in a more apposite coordinate system. Furthermore, the technique is also able to “self-diagnose” the characteristic time after which the approximation is guaranteed to be valid.

Many interesting open questions remain. For example, we do not yet know of any systematic way of investigating the significance of the order parameters with respect to the domain from which the model came. In other words, EAE returns some composite variables which are “natural” for the given problem and yet they may not correspond with any know “standard” entities in the field.

Moreover, although, in principle, we can see no impediment to applying EAE to higher dimensional systems, in practice, this will be a major coding effort. However, we are encouraged that our technique has the *promise* of scaling up whereas others (such as DAE, many current computational techniques and, in

fact, analytic approaches in general) do not. In order to make progress, it would be useful to have available a body of test models in their pre-approximated state.

As the EAE procedure is a generalisation of the widely used DAE technique, we believe that it holds the promise of becoming a genuinely useful tool in the repertoire of the model approximator.

6 Bibliography

[1] G. L. Baker & J. P. Gollub, *Chaotic Dynamics*, Cambridge University Press (1990)

[2] H. Haken, *Synergetics, An Introduction*, Springer-Verlag (1977)

[3] H. Haken, *Information & Self-Organization*, Springer-Verlag (1988)

[4] B. Kuipers, *Abstraction by Timescale in Qualitative Simulation*, in Proceedings of AAAI-87, pp621–625 (1987)

[5] R. de Mori & R. Prager, *Perturbation Analysis with Qualitative Models*, in Proceedings of IJCAI-89, pp1180–1186 (1989)

[6] O. Raiman, *Order Of Magnitude Reasoning*, Proceedings AAAI-86, pp105–112, Morgan-Kaufmann (1986)

[7] O. Raiman & B. Williams, *Caricatures: Generating Models of Dominant Behavior*, Xerox PARC Tech. Rep. (to appear) (1992)

[8] R. Rosen, *Dynamical System Theory in Biology, Vol. 1, Stability Theory & Its Applications*, Wiley-Interscience (1970)

[9] E. Sacks, *Piecewise Linear Reasoning*, in Proceedings AAAI-87, pp655–659, Morgan-Kaufmann (1987)

[10] E. Sacks, *Automatic Qualitative Analysis of Dynamic Systems Using Piecewise Linear Approximations*, Artificial Intelligence, 41, pp313–364 (1989)

[11] E. Sacks, *A Dynamic Systems Perspective on Qualitative Simulation*, Artificial Intelligence, 42, pp349–362 (1990)

[12] E. Sacks, *Automatic Analysis of One-Parameter Ordinary Differential Equations by Intelligent Numeric Simulation*, Artificial Intelligence, 48, pp27–56 (1991)

[13] R. Serra et al., *Introduction to the Physics of Complex Systems*, Pergamon Press (1986)

[14] D. Weld, *Comparative Analysis*, Artificial Intelligence, 36, pp333–373 (1988)

[15] D. Weld, *Choices for Comparative Analysis: DQ Analysis or Exaggeration*, Artificial Intelligence in Engineering, Vol. 3, #3 (1988)

[16] C. P. Williams, *Predicting the Approximate Functional Behaviour of Physical Systems*, Ph.D. thesis, Department of Artificial Intelligence, University of Edinburgh (1989)

[17] C. P. Williams, *Analytic Abduction from Qualitative Simulation*, to appear in Recent Advances in Qualitative Physics, B. Faltings & P. Struss (eds.), M.I.T. Press (1992)

[18] K. Yip, *Extracting Qualitative Dynamics from Numerical Experiments*, in Proceedings AAAI-87, Morgan-Kaufmann (1987)

[19] K. Yip, *Generating Global Behaviors Using Deep Knowledge of Local Dynamics*, in Proceedings AAAI-88, Morgan-Kaufmann (1988)

[20] K. Yip, *Understanding Complex Dynamics by Visual and Symbolic Reasoning*, Artificial Intelligence, 51, pp179–221 (1991)